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LOBACHEVSKIAN GEOMETRY

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ИЗДАТЕЛЬСТВО
МИР

ПОПУЛЯРНЫЕ ЛЕКЦИИ ПО МАТЕМАТИКЕ

А. С. Смогоржевский

О ГЕОМЕТРИИ ЛОБАЧЕВСКОГО

ИЗДАТЕЛЬСТВО «НАУКА» МОСКВА

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LOBACHEVSKIAN GEOMETRY

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AUTHOR'S NOTE

The aim of this book is to acquaint the reader with the fundamentals of Lobachevsky's non-Euclidean geometry.

The famous Russian mathematician N. I. Lobachevsky was an outstanding thinker, to whom is credited one of the greatest mathematical discoveries, the construction of an original geometric system distinct from Euclid's geometry. The reader will find a brief biography of N. I. Lobachevsky in Sec. 1.

Euclidean and Lobachevskian geometries have much in common, differing only in their definitions, theorems and formulas as regards the parallel-postulate. To clarify the reasons for these differences we must consider how the basic geometric concepts originated and developed, which is done in Sec. 2.

Apart from a knowledge of school plane geometry and trigonometry reading our pamphlet calls for a knowledge of the transformation known as inversion, the most important features of which are reviewed in Sec. 3. We hope that the reader will be able to grasp its principles with profit to himself and without great difficulty, since it, and Sec. 10, play very important, though ancillary, role in our exposition.

Section 1. A Brief Essay on the Life and Work of N. I. Lobachevsky

Nikolai Ivanovich Lobachevsky was born on December 1, 1792 (November 20 by the old Julian style), the son of an ill-paid civil servant. Early in their life Nikolai Lobachevsky and his two brothers were left in the sole care of their mother, an energetic and clever woman, who, despite her extremely meager means, sent them all to the Kazan grammar school.

Lobachevsky studied there from 1802 to 1807, at Kazan University from 1807 to 1811. Possessing brilliant mathematical talents he successfully completed the course of studies and after graduating remained at the University to work for a professorship, which was conferred on him in 1816.

Lobachevsky's teaching left a deep impress on the memories of his students. His lectures were noted for their lucidity and completeness of their exposition. His knowledge of various branches of science was extensive and many-sided, which enabled him to lecture not only on mathematical subjects but also on mechanics, physics, astronomy, geodesy, and topography.

Lobachevsky was elected rector of Kazan University in 1827 and occupied this post for nearly twenty years. Being a talented and energetic administrator, with a good insight into the aims of higher education, he succeeded in making Kazan University a model higher educational institution of his time. On his initiative the university began publishing *Scientific Proceedings*. Under him construction of the university's buildings was broadly developed, an astronomical observatory founded.

But it was his scientific work that brought Lobachevsky world fame. He immortalized his name by creating the non-Euclidean geometry now called after him[1].

On 23 (11) February, 1826 Lobachevsky read a paper at a meeting of the Department of Physico-mathematical Sciences of Kazan University in which he first communicated the non-Euclidean geometry discovered by him. The first published presentation of its principles was his memoirs *On the Fundamentals of Geometry* published in 1829 and 1830 in the journal *Kazan Herald.*

Most of Lobachevsky's contemporaries did not understand his discovery, and his works on geometry had a hostile reception

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[1] Its other name – hyperbolic geometry – is due to the fact that in it a straight line like a hyperbola in Euclidean geometry has two infinitely removed points (see Sec. 4).

both in Russia and abroad. His ideas were too daring and departed too far from the notions that then dominated science so that much time had to pass before they won general recognition, which came only after his death.

Lobachevsky was not dissuaded of the correctness of his conclusions by his critics' attacks and continued, with his native energy and determination, to work on the development of his geometric system, publishing a number of works devoted to problems of non-Euclidean geometry. The last of them, completed by Lobachevsky not long before his death, had to be dictated as he himself was unable to write any more because of the blindness that affected him in his last years.

Lobachevsky's scientific activity was not restricted to investigation in geometry: he also made several fundamental contributions to algebra and calculus. The method of approximate solution of algebraic equations he worked out is very elegant and efficient.

Lobachevsky's philosophical views had a clearly expressed materialist slant. He considered experiment and practice the most reliable means of testing theoretical conclusions. He demanded teaching of mathematics such as would bring out the real phenomena behind mathematical operations.

In 1846 Lobachevsky was relieved of his duties at the University and appointed assistant trustee of the Kazan educational district.

He died on 24 (12) February, 1856. In 1896 a monument was erected in Kazan to honour his memory[1].

Section 2. On the Origin of Axioms and Their Role in Geometry

To elucidate the role of axioms let us trace the general outline of the most important steps in the development of geometry from ancient times.

The birthplace of geometry was the countries of the Ancient East where important practical rules for measuring angles, areas of certain figures, and the volumes of the simplest solids were worked out thousands of years ago to meet the needs of land mensuration, architecture and astronomy. These rules were developed empirically (from experience) and appear to have been passed on by word of mouth: in the oldest texts that have come down to us we often come across applications of geometric rules but find no attempts to formulate them.

In the course of time the circle of the objects to which the geometric knowledge acquired was applied broadened, and a need arose to formulate the rules, in the most general form possible, which brought about a transition in geometry from concrete notions to abstract concepts. For example, the rule developed for measuring the area of a rectangular plot of land proved applicable to measuring the area of a carpet, the surface of a wall, etc., and as a result an abstract concept, a rectangle, arose.

So a system of knowledge was formed which came to be termed geometry. At its early stage it was an empirical science, i. e. one in which all the results were derived directly from experience.

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[1] The reader can find more details on Lobachevsky s life in

V. F. Kagan, *N. Lobachevsky and His Contribution to the World Science* by the Foreign Languages Publishing House, Moscow, 1957;

P. A. Shirokov and V. F. Kagan, *Structure of Non-Euclidean Geometry.* Issue 1 of the series *Lobachevsky's Geometry and Development of Its Ideas,* Moscow-Leningrad, 1950 (in Russian). One chapter of this book contains a brief and skilful presentation of Lobachevskian geometry understandable to a wide range of readers.

The development of geometry took a new direction when it was noticed that some of its propositions did not need empirical substantiation since they could be deduced from its other propositions as conclusions following from the laws of logic. The propositions of geometry were now divided into two classes: those established empirically (later called axioms) and those provable logically on the basis of axioms (theorems).

Because logical substantiation, which does not require either special instruments or numerous tiresome measurements, is technically much simpler than the empirical approach, the scientists of the antiquity were naturally faced with the problem of reducing to a minimum the number of propositions of the first type (axioms) so as to lighten the geometer's job by shifting its main load to the sphere of logical reasoning. This goal proved attainable since geometry was abstracted from all properties of bodies except extension, a most essential one but so simple that all possible geometric relationships can be deduced by the laws of logic from a limited number of premises or axioms.

Thus geometry was converted from an empirical science into a deductive one[1] with its present-day axiomatic presentation.

The earliest book that has come down to us with a systematic exposition of the main propositions of geometry was Euclid's *Elements* written around 300 B.C. This work has the following structure: after definitions and axioms come the proofs of theorems and solutions of problems; each new theorem being proved on the basis of axioms and previously proved theorems. The axioms are not proved but simply stated.

For two thousand years Euclid's *Elements* enjoyed undisputed authority among scholars. But one point in it did not seem quite justified. That was the parallel-postulate stated as follows:

If a straight line falling on two straight lines makes the two interior angles on the same side of it taken together less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles together less than two right angles[2].

---

[1] A deduction is the deriving of a conclusion. A science is called deductive when its new statements are deduced from preceding ones by way of logic.

[2] In school textbooks Euclid's parallel-postulate is replaced by the following equivalent proposition: *Only one straight line can be drawn parallel to a given straight line through a point not on this line.*

Two axioms of Euclidean or any other geometry are considered equivalent when the same conclusions follow from both, provided all the other axioms of the geometry remain valid.

The validity of Euclid's axiom of parallels aroused no doubts. The uncertainty regarding it lay in something else: was it justifiable to place it among the axioms? could it not be proved from the axioms of his *Elements* and so transferred to the category of a theorem?

Initially the attempts to prove the parallel-postulate reflected the tendency mentioned above to reduce the number of geometric propositions requiring empirical substantiation. In the course of time the situation has changed: the empirical origin of the axioms was forgotten and they came to be treated as self-evident truths, independent of any experience or experiment whatsoever[1]. This view gave rise to the conviction that the parallel-postulate, which it is difficult to recognize as self-evident because of its complexity, was not in fact an axiom and so the statement affirmed in it could be proved. But the many efforts in this direction did not produce positive results; like an enchanted treasure, the parallel-postulate would not yield up its secrets to investigators. The attempts to prove it, which consumed a tremendous amount of mental effort by generations of scholars, failed as the price of idealistic interpretation of the essence of axioms.

The most common type of erroneous proof of Euclid's parallel-postulate was to replace it by an equivalent proposition, for instance: *a perpendicular and an oblique line on one and the same straight line intersect*; or, *there is a triangle similar to a given triangle but not equal to it*; or: *the locus of points equidistant from a given straight line and located on one side of it, is a straight line*; or, *any three points are either collinear or cocyclic*. Later we shall demonstrate that all these propositions are fallacious if Euclid's axiom of parallels does not hold. Consequently, by taking any of them as an axiom we thereby assume the validity of Euclid's parallel-postulate, i. e. assume to be correct what we want to prove.

Lobachevsky took a different path in his investigations in the theory of parallels. Having started with attempts to prove the axiom of parallels he soon noticed that one of them led to quite unexpected results. This attempt consisted in using proof by contradiction (reductio ad absurdum) and was based on the following argument: if Euclid's parallel-postulate is the consequence of the other postulates of *Elements*, and if one assumes, in spite of it, that *at least two straight lines not intersecting a given straight line can be drawn through a point located outside of that line in the same plane as it*, then this assumption should lead sooner or later, in its immediate or remote consequences, to a contradiction. But in considering more and more new consequences of this assumption, Lobachevsky became convinced that, no matter how paradoxical they seemed from the standpoint of Euclidean geometry, they formed a consistent system of theorems that could form the basis of a new scientific theory.

---

[1] It is known that persons who were born blind but who have had their eyesight restored surgically, cannot distinguish a cube from a sphere for some time after the operation without first touching them. This demonstrates a need of experience for the correct perception of geometrical images, without which geometric concepts cannot be formed.

Thus the foundation of non-Euclidean geometry[1] was laid; its axiom of parallels differs from the Euclidean and coincides with the assumption given above which we shall refer to henceafter as Lobachevsky's parallel-postulate.

But it still remained obscure whether it could confidently be stated that not one of the infinite set of possible consequences of Lobachevsky's parallel-postulate would lead to a contradiction. Lobachevsky outlined a way to solve this problem, pointing out that the consistency of the geometry discovered by him should follow from the possibility of arithmetizing it, i. e. from the possibility to reduce the solution of any geometric problem to arithmetic calculations and analytic transformations by using the formulas of the hyperbolic trigonometry derived by himself. Other scientists later found rigorous proofs of the consistency of his geometry.

Lobachevsky's investigations in the domain of the hyperbolic geometry were very wide covering its elements, trigonometry, analytical geometry and differential geometry. Using the methods of his geometry he derived more than 200 new formulas for calculating definite integrals.

Lobachevsky's discovery was considered by his contemporaries and even by his pupils as monstrous nonsense, insolent defiance of logic and common sense[2]. Such an attitude toward a great idea demolishing hallowed conceptions is not surprising. Copernicus's heliocentric theory, which denied what seemed completely obvious and asserted what seemed unthinkable, had just as hostile a reception. It needed very profound considerations to understand the admissibility of two different geometries. Let us now turn to presentation of some of the most easily understood arguments.

---

[1] It has since been found that, apart from the geometry discovered by Lobachevsky, many other non-Euclidean geometries can be constructed.

[2] One cannot, of course, groundlessly suspect Lobachevsky's contemporaries of being unable to understand his discovery: many did not express any opinion, possibly because the range of their scientific interests did not include the sphere of Lobachevsky's investigations; we also know that the famous German mathematician Karl Gauss and the outstanding Hungarian geometer János Bolyai, who, independently of Lobachevsky, came to the idea of the possibility of constructing a non-Euclidean geometry, shared his views. But Gauss, fearing not being understood and ridiculed, did not publish any support of Lobachevsky's

The section on plane geometry in school textbooks studies a plane independently of the surrounding space; in other words, planimetry is the geometry of a Euclidean plane. Geometries of certain curvilinear surfaces are also well known; an example is spherical geometry, which is widely used in astronomy and other branches of knowledge.

In every science the simplest concepts are most important. In Euclidean geometry these are the concepts of point, straight line, and plane. These terms are retained in non-Euclidean geometries, so that by a "straight line" is meant a line along which the shortest distance is measured between two points; a "plane" is a surface possessing the property such that if two points of a "straight line" belong to the surface, then all the points of that "straight line" belong to the surface. In spherical geometry, for instance, a sphere and its great circles are referred to, respectively, as a "plane" and "straight lines". This terminology is quite appropriate since, in any geometry, a "straight line" is the simplest of all lines and a "plane" is the simplest of all surfaces, the former possessing the most important property of the Euclidean straight line and the latter of the Euclidean plane[1].

Let us note certain features of spherical geometry. For illustrative purposes we shall consider it as the geometry of the surface of a globe. It is not difficult to grasp that two "straight lines" in this geometry (e. g. two meridians) always intersect at two diametrically opposite points on the globe. Furthermore, the sum of the angles of a spherical triangle is greater than π; for example, in a triangle bounded by a quarter of the equator and by the arcs of two meridians (Fig. 1) all three angles are right angles[2].

---

ideas, and Bolyai, seeing that his own investigations in non-Euclidean geometry (published in 1832) had not received recognition, abandoned his mathematical studies. Thus Lobachevsky was left alone to struggle for the correctness of his ideas.

[1] It should be noted that in projective geometry there is no concept of the distance between two points; the interpretation of the concepts of a "straight line" and of a "plane" does not apply in such geometries

[2] The angle between two lines at their point of intersection is defined as the angle between their tangents at this point.

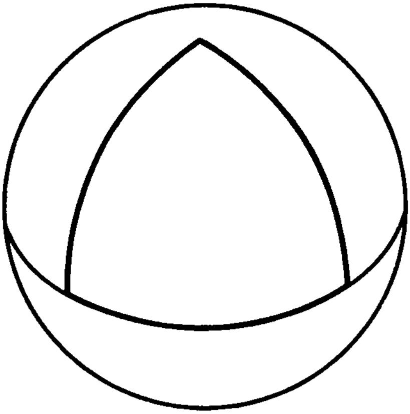

Fig 1

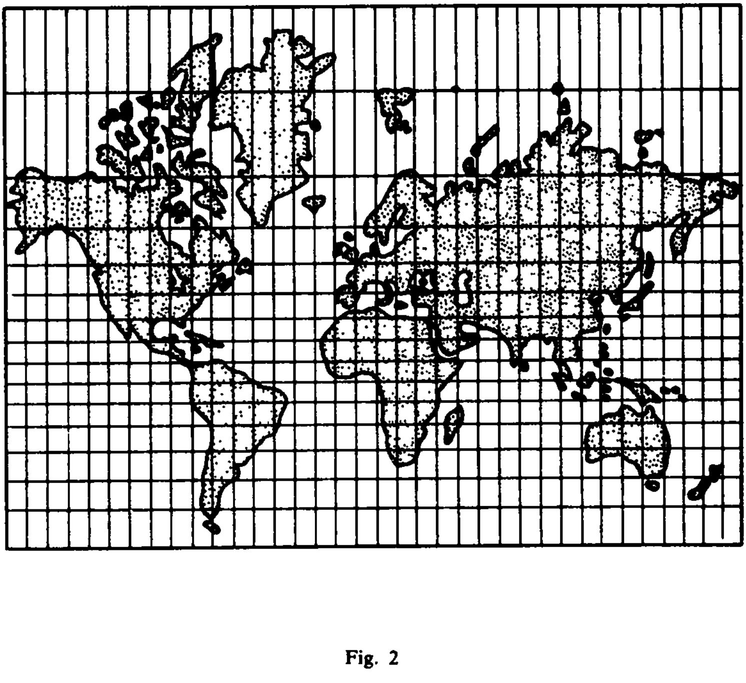

Fig. 2

Apart from globes, of course, maps of the terrestrial surface are used in geography. This is equivalent to studying spherical geometry by considering maps of a sphere, which is quite possible provided that it is indicated how to measure the actual lines and the actual angles between them from their representations on the map, for the latter are distorted and the character of the distortion is not uniform over the whole map. On maps of the Earth's surface employing Mercator's projection[1] (Fig. 2), for example, the meridians are projected as parallel lines, and the perpendiculars, which correspond to the geographic parallels, are such that a segment representing 1° of a parallel has the same length irrespective of the latitude; but in reality the length of one degree of a parallel is the shorter the higher the latitude.

Since a surface has two dimensions, the geometry studying figures lying on a certain surface is usually called two-dimensional, and the surface itself a two-dimensional space. Two types of two-dimensional geometry have been known since antiquity, Euclidean (for a plane) and spherical. Mathematicians did not assign special importance to the existence of a two-dimensional non-Euclidean geometry, namely spherical geometry, for the simple reason that the sphere was studied in three-dimensional Euclidean space, which made them disregard the non-Euclidean properties of the sphere as such.

As a result of Lobachevsky's investigations it was realized that not only are surfaces with non-Euclidean properties imaginable but also three-dimensional non-Euclidean spaces.

The introduction of the concept of three-dimensional non-Euclidean geometries may puzzle unless we give the following explanation.

It is sometimes convenient to represent the results of studying a certain class of phenomena in a geometrical form. Data on the growth of labour productivity, for example, are often shown in the form of graphs and diagrams. This demonstrates that various real processes and states with no direct connection with geometry can be depicted by means of geometric images.

If a graph is considered as a line on a Euclidean plane it becomes clear that images of the two-dimensional Euclidean geometry are employed in our example. In more complicated cases we may have to resort to three-dimensional and even

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[1] Gerhard Mercator (1512–1594) was an outstanding Flemish cartographer The projection proposed by him in 1569 became universally accepted and the nautical charts have been compiled by it ever since

multi-dimensional Euclidean and non-Euclidean geometries. But it does not follow that they all describe the relations in extension; there are theories that employ geometric terms in their formulations and these terms, generally speaking, are assigned meanings that are not related to spatial concepts. Thus by adding time as a fourth dimension to the three dimensions of real space we introduce the concept of four-dimensional space in which a given time interval is considered as a "segment of a straight line". In most cases this approach only creates an appearance of vizualization, nevertheless it facilitates the analysis of phenomena to a certain extent when they are studied by this method.

So the construction of non-Euclidean geometries is justified by the possibility of applying their conclusions to actually existing objects. The fact that these conclusions are expressed in terms of geometry is of no real consequence; it is not hard to modify the geometric formulations so that they correspond to the properties of the objects and phenomena in question.

The substitution of certain concepts for others, let us note, is a common practice in applied mathematics when a theory describes qualitatively different objects governed by the same mathematical laws[1].

Three-dimensional geometries call for special attention. Irrespective of their other applications they can be regarded as hypotheses claiming to describe the properties of real space. Which one corresponds most closely to reality is a problem that can only be solved by experimental testing.

But let us note the following fact, important for our further exposition: a map of a Lobachevskian plane can be constructed on a Euclidean plane, and in more than one way, just as is done for a sphere. We shall use analysis of such a map as the basis for our study here of hyperbolic geometry.

Lobachevsky's geomerty received general recognition in the following circumstances. In 1868 the Italian geometer Eugenio (1835-1900) Beltrami has discovered that there was a surface in the Euclidean space that possessed the properties of a Lobachevskian plane, or rather of a certain segment of this plane (if the shortest lines on the surface are considered as "straight lines"). This discovery, which soon led to the construction of various maps of the Lobachevskian plane, convinced scientists of the correctness of the Russian geometer's ideas, gave an impetus to a deeper study of his work, and stimulated the starting of many investigations in the field of non-Euclidean geometries.

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[1] As to the practical application of this principle see the section on simulation in V. G. Boltyansky's *Differentiation Explained* (Mir Publishers, Moscow).

The discovery of non-Euclidean geometries posed an extremely complicated problem to physics, that of explaining whether real physical space was Euclidean as had earlier been believed, and, if it was not, to what type of non-Euclidean spaces it belonged[1]. To answer it, it is necessary to check the validity of the axioms experimentally, it being clear that, with the improvement in measuring instruments, the reliability of the experimental data obtained will increase and with it the possibilities of penetrating into details that earlier escaped investigators' attention.

Thus Lobachevsky brought geometry back to a materialist interpretation of its axioms as propositions postulating the basic geometric properties of space, perceived by humanity as the result of experience.

We still cannot consider the problem of the geometric structure of real physical space completely resolved. Nevertheless we may note that in the modern theory of relativity real space is considered on the basis of numerous data to be non-Euclidean, and to have geometric properties more complex than those of Lobachevskian space. One of the heaviest blows to belief in the Euclidean structure of real space was dealt by the discovery of the physical law that there can be no velocity exceeding the velocity of light.

Now we can answer the question one hears fairly often, namely, which of the two geometries, Euclid's or Lobachevsky's is the true one.

No similar question arises regarding the two-dimensional Euclidean and spherical geometries; both are obviously true, but each has its own sphere of application. The formulas of spherical geometry cannot be used for plane figures nor those of two-dimensional Euclidean geometry for figures on a sphere. The same is true of the different three-dimensional geometries: each of them, being logically consistent, has its application in a certain field, not necessarily geometrical in character; but it would be invalidated if we ascribed it a universal character.

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[1] When this problem is considered, the possibility of real space being non-uniform must not be neglected, that is to say, the possibility that its geometric structure may be different at different points.

As to the geometric structure of real space, the problem, as we have indicated, comes within the domain of physics and cannot be resolved by means of pure geometry. Its specific feature, by the way, is that no geometry represents spatial relations with absolute accuracy; the molecular structure of matter, for example, precludes the existence of solids of dimensions perceivable by touch that would have the geometric properties of an ideal sphere. Therefore the application of geometric rules to the solution of concrete problems inevitably produces only approximate results. So our concept of the geometric structure of real space boils down to a scientifically justified conviction that one geometry provides a better description of actual spatial relations than others.

Though the theory of relativity uses the formulas of non-Euclidean geometry, it does not follow that Euclid's geometry must be discarded, as happened to astrology, alchemy and to pseudo-sciences like them. Both geometries are tools for investigating spatial forms but the non-Euclidean enables finer studies to be made while Euclid's is adequate for solving most practically important problems with a very high degree of accuracy; and since it is, at the same time, characterized by great simplicity, its wide application is always permanently guaranteed.

To conclude this brief outline let us note the new ideas introduced by Lobachevsky into the development of geometry.

The scientific contributions of this outstanding thinker were not restricted to his unveiling of the thousand-year-old mystery of the axiom of parallels; the significance of his work was immeasurably greater.

By subjecting one of Euclid's axioms to critical analysis, Lobachevsky laid the basis for reconsideration of some of initial propositions of the Euclidean system, which subsequently led to the development of rigorous scientific principles for the axiomatic construction of geometry and other branches of mathematics.

Lobachevsky's discovery of hyperbolic geometry freed the science of spatial forms from the narrow framework of the Euclidean system. His geometry found direct application in the theory of definite integrals and in other spheres of mathematics.

Lobachevsky initiated the treatment of problems that could not have arisen in the former state of mathematics, including that of the geometric structure of real space. Without it the theory of relativity, one of the greatest achievements of modern physics, could not have been developed. Taking Lobachevsky's investigation as a start, scientists have built a theory that makes it possible to analyse the processes taking place inside the atomic nucleus.

In conclusion let us note the gnoseological significance[1] of the ideas of this great Russian mathematician. Before Lobachevsky geometry was dominated for centuries by the idealistic view originating with the Greek philosopher Plato. By ascribing the axioms of the Euclidean system an absolute character, Plato denied their empirical origin. Lobachevsky decisively shattered this outlook and returned geometry to a materialist position.

Section 3. Inversion

Suppose there to be a rule that allows the transition from any given figure to another in such a way that the second is completely defined once the first is fixed, and vice versa. The transition so made is called a geometric transformation. The most commonly used geometric transformations are parallel translation, similarity transformation, rotation of a figure, projection, and inversion. Inversion is used extensively in mathematics, for example, as a method of solving problems of construction, in the theory of functions of a complex variable, and in studying maps of a Lobachevskian plane.

In this section we will define inversion and its related concepts, and consider a number of its basic properties.

Let a circle k be drawn in a plane α, with a radius r and a centre O, and a point A, not identical with the point O. Let us take a point A' on the ray OA in such a way that the product of the segments OA and OA' equals the square of the radius of the circle k:

$$OA \cdot OA' = r^2 \tag{1}$$

Let us agree to call points A and A' **symmetrical** with respect to the circle k.

If either of the points A and A' is located outside the circle k, then the second point lies within it, and vice versa; for example, from the inequality $OA > r$ we conclude, taking condition (1) into account, that $OA' < r$. But if either A or A' lies on the circle k then A and A' coincide.

Consider Fig. 3 in which AB is a tangent to the circle k, and BA' is a perpendicular to OA. Since OA' is the projection of

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[1] Gnoseology—the science of cognition.

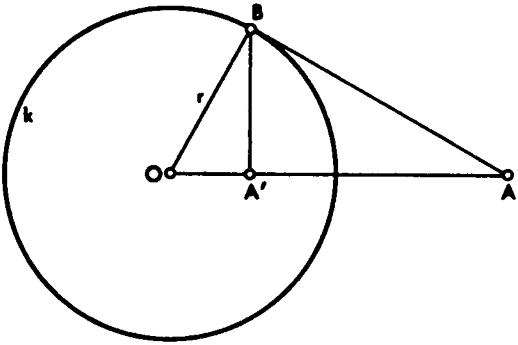

Fig. 3

the leg OB of the right-angled triangle OAB on the hypotenuse OA,

$$OA \cdot OA' = OB^2 = r^2$$

points A and A' are therefore symmetrical with respect to k. Here clearly is a means of constructing the point A' when the point A is given, and the point A when the point A' is given.

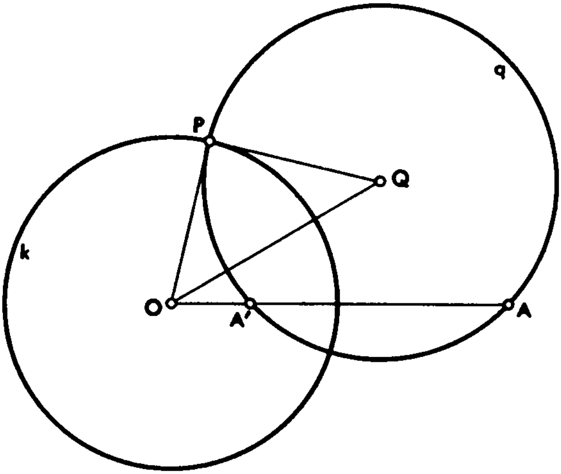

Fig. 4

Theorem 1. *If a circle q passes through two distinct points A and A', symmetrical with respect to a circle k, then the circles k and q are mutually orthogonal.*

Two circles are called mutually orthogonal if they intersect at right angles, i. e. if the tangents to them at the point of intersection (or, what is the same thing, if their radii drawn at this point) are mutually perpendicular.

Let P be one of the points of intersection of the circles k and q (Fig. 4). Since OP is the radius of k, equation (1) takes the form $OA \cdot OA' = OP^2$. On the other hand, the product of segments OA and OA' is equal to the square of the tangent to q drawn at point O; hence OP is tangential to q. Consequently, radii OP and QP of these circles are mutually perpendicular, and the circles are mutually orthogonal.

Note that any circle passing through two distinct points, symmetrical with respect to a straight line, intersects the line at right angles. The analogy between this property and the fact established by Theorem 1 has induced the transfer of the term "symmetry" to the case of two points so situated with respect to a given circle that any circle passing through them is orthogonal to the given circle.

Theorem 2. *If the circles k and q are mutually orthogonal, then a straight line passing through the centre O of k and intersecting q, does so at points symmetrical with respect to k.*

Let us denote the points where the straight line intersects q by A and A' and one of the points in common of the two circles k and q by P (Fig. 4). Since the circles are mutually orthogonal, line OP is a tangent to q; therefore, $OA \cdot OA' = OP^2$. Hence we conclude that points A and A' are symmetrical with respect to k.

Theorem 3. *Given a triangle OAB in which O is the centre of a circle k, and points A' and B' are symmetrical to A and B with respect to k. Then*

$$\angle OAB = \angle OB'A' \quad \text{and} \quad \angle OBA = \angle OA'B'$$

Consider Fig. 5. From the equation

$$OA \cdot OA' = OB \cdot OB'$$

which follows from condition (1), we get $OA : OB' = OB : OA'$. Therefore, triangles OAB and $OB'A'$, possessing a common angle AOB, are similar. Hence we conclude that the theorem is valid.

Note that a circle can be drawn about the quadrangle $ABB'A'$,

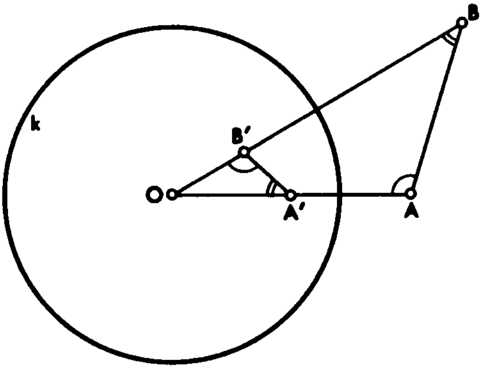

Fig. 5

since $\angle A'AB + \angle A'B'B = \pi$. As follows from Theorem 1, this circle is orthogonal to k.

Now let us consider a transformation of the plane α consisting in the following: any two points on this plane, symmetrical with respect to the circle k, change places. This transformation is called **inversion**, the circle k is called the **circle of inversion**, and its centre the **pole of inversion**. If inversion with respect to k transforms a figure F into F', we say that F is symmetrical to F', and F' to F, with respect to k.

Note that there is no point that is symmetrical to the pole of inversion with respect to the circle of inversion.

It will easily be seen that points outside the area bounded by the circle of inversion are transformed into the points inside it, excluding the inversion pole, and vice versa; the points of the circle of inversion are transformed into themselves; a straight line, drawn through the pole of inversion O, is transformed into itself but in the process loses point O.

Theorem 4. *A straight line not passing through the pole of inversion is transformed by inversion into a circle passing through the inversion pole.*

Let A be the foot of the perpendicular dropped from the pole of inversion O to a straight line l, B an arbitrary point of the line l, A' and B' two points respectively symmetrical to points A and B with respect to the circle of inversion k (Fig. 6). Let us construct a circle q with the segment OA' as its diameter. By Theorem 3, $\angle OB'A' = \angle OAB$, therefore $\angle OB'A' = \pi/2$; hence, point B' lies on the circle q. On the other hand, let C' be any point, distinct from O, of the circle q; then the straight

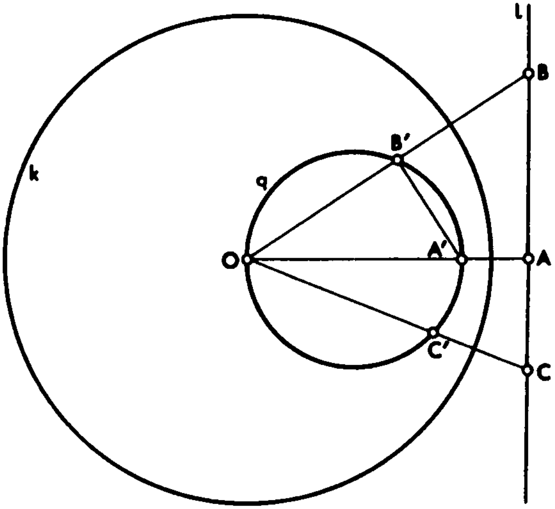

Fig. 6

line OC' will intersect l at a certain point C, which is transformed by this inversion, as will readily be seen, into point C'. The theorem is thus proved, but we must remember that the line l is transformed into a figure consisting of the circle q **without point** O.

Note that the centre of the circle q lies on the perpendicular dropped from O to l.

If the straight line l has no points in common with the circle of inversion k, then q lies inside k.

If l is tangential to k at a certain point, then q is tangential to k at the same point.

If l and k intersect, then q passes through their points of intersection.

Theorem 5. *Inversion transforms a circle passing through the pole of inversion into a straight line not passing through the pole of inversion.*

Let O (the pole of inversion), A and B be three distinct points on the circle q, and A' and B'. two points symmetrical to A and B with respect to the circle of inversion. By virtue of Theorem 4, the line $A'B'$ is transformed into a circle passing through O, A and B, i. e. into the circle q; hence it follows that q is transformed into the straight line $A'B'$.

Theorem 6. *Inversion transforms a circle not passing through the pole of inversion into a circle that likewise does not pass through the inversion pole.*

Let k be a circle of inversion with radius r and its centre at O, and q a given circle not passing through O (Fig. 7). Now take an arbitrary point A on q and denote the second point of intersection of the straight line OA and q by B, and the points symmetrical to A and B with respect to k respectively by A' and B'. Then

$$OA \cdot OA' = OB \cdot OB' = r^2$$

Whence

$$\frac{OA}{OB'} = \frac{OB}{OA'} \tag{2}$$

and

$$OA \cdot OB \cdot OA' \cdot OB' = r^4$$

The product

$$OA \cdot OB = g$$

is not altered, by virtue of the well-known theorems of elementary geometry, when point A is displaced along q. Therefore g is a constant quantity, positive when O lies outside q, and negative when O is inside q (since, in the second case, the directions of the segments OA and OB are opposing).

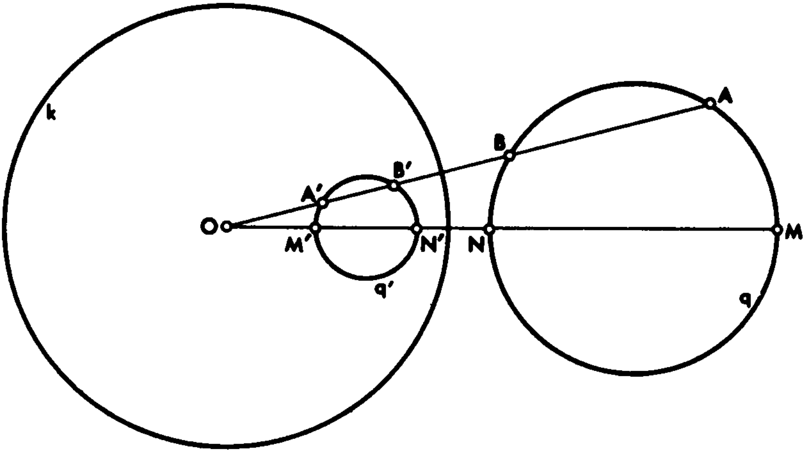

Fig. 7

From the last two equations we derive $OA' \cdot OB' = \frac{r^4}{g}$, therefore

$$\frac{OA}{OB'} \cdot \frac{OB}{OA'} = \frac{g^2}{r^4}$$

and, taking Eq. (2) into account,

$$\frac{OA}{OB'} = \frac{g}{r^2}$$

(the sign is selected correctly since the segments OB and OB' have the same direction). As follows from the last equation the figures traced by points A and B' are similar, consequently the theorem is proved: the point B' traces a circle (which we shall denote by q').

The pole of inversion O will be the centre of similarity of circles q and q', exterior if $g > 0$, and interior if $g < 0$. In the first case O lies outside, and in the second case inside the circles q and q'.

If q is tangential to k at a certain point, then q' is tangential to k at the same point.

If k and q intersect, then q' passes through their points of intersection.

A circle q orthogonal to k is transformed into itself by inversion with respect to k (q' coincides with q), which follows from Theorem 2.

If the line through the centres of the circles k and q cuts q at the points M and N, then the segment $M'N'$ (where M' and N' are points symmetrical to M and N with respect to k) will be the diameter of the circle q' (Fig. 7). This note can be used to construct the circle q'.

Note that the centres of the circles q and q' are not symmetrical with respect to the circle of inversion k.

Theorem 7. *The points of intersection of two circles p and q, orthogonal to the circle k, are symmetrical with respect to k.*

The theorem is obvious since each of the circles p and q is transformed into itself by inversion with respect to k; consequently, their points of intersection A and A_1 will change places (Fig. 8).

Theorem 8. *If M and M' are two points, symmetrical with respect to a circle k on two curves m and m', also symmetrical with respect to k, then the tangents to m and m' at points M and M' are either perpendicular to the straight line MM' or form with this line an isosceles triangle with the base MM'.*

Take on m a point N, distinct from M, and plot a point N' symmetrical to N with respect to k (Fig. 9). It is obvious that N' lies on m'. The lines MM' and NN' pass through the

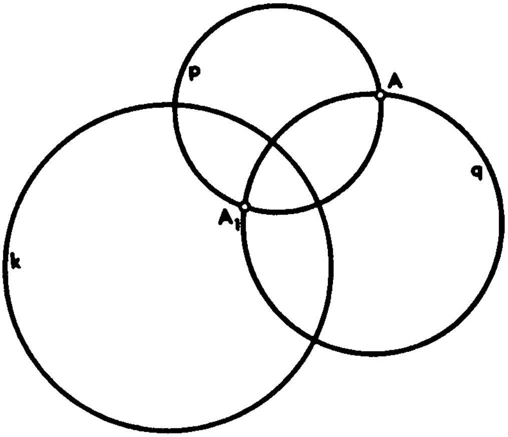

Fig. 8

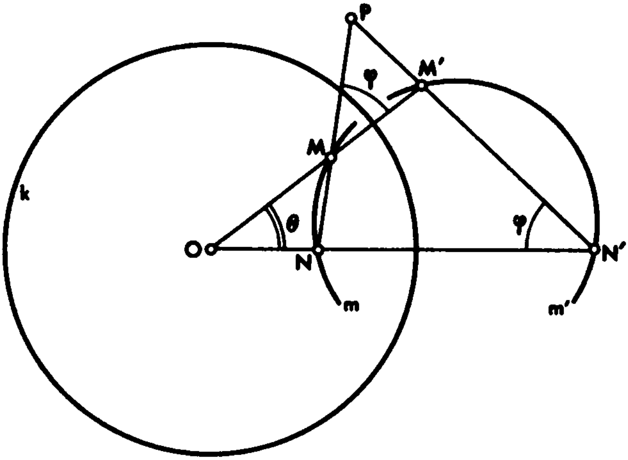

Fig 9

centre O of the circle k. Construct straight lines MN and $M'N'$, denoting their point of intersection by P. If

$$\angle MON = \theta, \quad \angle OMN = \varphi$$

then, by virtue of Theorem 3, $\angle ON'M' = \varphi$. Therefore, in the triangle $MM'P$

$$\angle M = \varphi, \quad \angle M' = \varphi + \theta$$

Let the angle θ approach zero under the condition that the point M is stationary. Then in the limiting configuration the secants MN and $M'N'$ will be transformed into tangents to m and m' at points M and M', and the triangle $MM'P$ will become isosceles. Indeed

$$\lim_{\theta \to 0} (\varphi + \theta) = \lim_{\theta \to 0} \varphi + \lim_{\theta \to 0} \theta = \lim_{\theta \to 0} \varphi$$

Thus the theorem is proved.

Theorem 9. *Inversion leaves angles unaltered.*

Let us consider the lines m and n intersecting at point A. Let inversion with respect to the circle k transform m, n and A into m', n' and A'. As follows from Theorem 8, the angle between the tangents to m and n at point A is equal to that between the tangents to m' and n' at point A'. Q.E.D.

A transformation that does not alter the angles is called **conformal.** It follows from the foregoing that inversion is a conformal transformation.

Section 4. Map of a Lobachevskian Plane

Consider a plane ω and a straight line u on it dividing ω into half-planes τ and τ'. Assume the half-plane τ to be the map of a two-dimensional space H We shall distinguish a length s of the line in H and a length σ of its image on the map· we shall call the quantities s and σ respectively the **hyperbolic** and **Euclidean** lengths.

The following principles will be used to measure lengths on our map:

1°. The hyperbolic length of a segment MN, parallel to the straight line u and located at a distance y from it, is equal to $\dfrac{MN}{y}$, i. e. to the ratio of the Euclidean length of this segment to its Euclidean distance from u.

2°. If σ is the Euclidean length and s the hyperbolic length of an arc of a curve (or of a segment of a straight line, not parallel to u), and y and y' are the minimum and maximum

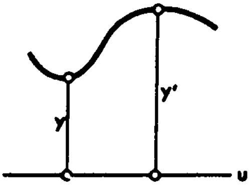

Fig. 10

Euclidean distances of its points from u, and if $y \neq 0$ (Fig. 10), then the following inequalities hold:

$$\frac{\sigma}{y'} < s < \frac{\sigma}{y}$$

We shall demonstrate later on that the space H whose map possesses the foregoing properties is a Lobachevskian plane.

On the basis of the principles 1° and 2° it is not difficult to indicate a general method for measuring hyperbolic lengths.

Let us first find the hyperbolic length s of an arc AB that has the following properties: if a point moves along this curve from A towards B, then its distance from the line u increases; the distance of the point A from u is not equal to zero; and the arc AB is smooth, i.e. has no sharp bends (Fig. 11).

Let us plot on the arc AB, proceeding from A to B, the points

$$A,\ P_1,\ P_2,\ \ldots,\ P_{n-1},\ B \tag{*}$$

Let the symbols

$$y_0,\ y_1,\ y_2,\ \ldots,\ y_{n-1},\ y_n$$
$$\sigma_1,\ \sigma_2,\ \ldots,\ \sigma_n$$
$$\zeta_1,\ \zeta_2,\ \ldots,\ \zeta_n$$

respectively denote: the Euclidean distances of points (*) from the straight line u; the Euclidean lengths of the arcs AP_1, P_1P_2,... ..., $P_{n-1}B$ constituting the segments of the arc AB; and the Euclidean lengths of the chords subtending these arcs.

Compile the sums:

$$\sum = \frac{\sigma_1}{y_1} + \frac{\sigma_2}{y_2} + \ldots + \frac{\sigma_n}{y_n}$$

$$\sum{}' = \frac{\sigma_1}{y_0} + \frac{\sigma_2}{y_1} + \ldots + \frac{\sigma_n}{y_{n-1}}$$

$$Z = \frac{\zeta_1}{y_1} + \frac{\zeta_2}{y_2} + \ldots + \frac{\zeta_n}{y_n}$$

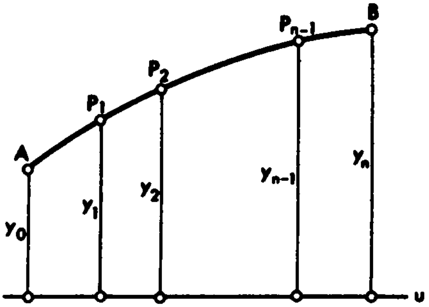

Fig. 11

By virtue of the principle 2° we obtain

$$\Sigma < s < \Sigma' \tag{3}$$

since according to the initial condition $0 < y_0 < y_1 < \ldots < y_n$. Now let us consider the difference

$$\Sigma' - \Sigma = \frac{\sigma_1}{y_0 y_1}(y_1 - y_0) + \frac{\sigma_2}{y_1 y_2}(y_2 - y_1) + \ldots + \frac{\sigma_n}{y_{n-1} y_n}(y_n - y_{n-1})$$

The right-hand side of this equation will increase if each of the quantities $\sigma_1, \sigma_2, \ldots, \sigma_n$ is replaced by the maximal of these quantities (denoted by σ') and each denominator is replaced by y_0^2. Consequently

$$\Sigma' - \Sigma < \frac{\sigma'}{y_0^2}(y_1 - y_0 + y_2 - y_1 + \ldots + y_n - y_{n-1}) = \frac{\sigma'}{y_0^2}(y_n - y_0)$$

If σ' tends to zero, it follows that the difference $\Sigma' - \Sigma$ also tends to zero.

Let us transform the sum Z to the form

$$Z = \frac{\sigma_1}{y_1} \cdot \frac{\zeta_1}{\sigma_1} + \frac{\sigma_2}{y_2} \cdot \frac{\zeta_2}{\sigma_2} + \ldots + \frac{\sigma_n}{y_n} \cdot \frac{\zeta_n}{\sigma_n}$$

Denoting the minimal of the ratios $\frac{\zeta_1}{\sigma_1}, \frac{\zeta_2}{\sigma_2}, \ldots, \frac{\zeta_n}{\sigma_n}$ by α and the maximal by β we obtain

$$\alpha\Sigma \leqslant Z \leqslant \beta\Sigma \tag{4}$$

Let the number n increase without limit and, at the same time, let each of the quantities $\sigma_1, \sigma_2, \ldots, \sigma_n$, and that also means σ', tend to zero. Then, as was proved above, the difference $\Sigma' - \Sigma$ will tend to zero while the quantities α and β will tend

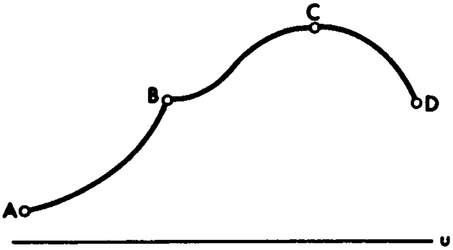

Fig. 12

to unity[1]. In this connection it follows from inequalities (3) and (4) that each of the sums Σ, Σ', Z will tend to the same limiting value and that this limit is equal to the hyperbolic length s of the arc AB.

It is more convenient to use the sum Z since it comprises the lengths of the Euclidean segments and not those of the arcs.

Thus

$$s = \lim Z = \lim \left(\frac{\zeta_1}{y_1} + \frac{\zeta_2}{y_2} + \dots + \frac{\zeta_n}{y_n} \right) \tag{5}$$

in which the passage to the limit is performed under the above-mentioned conditions.

Let us note that y_1 in equation (5) can be taken as equal to the distance of an arbitrary point on the segment AP_1 from the line u, y_2 as equal to the distance of an arbitrary point on the segment P_1P_2 from u, and so on, which can alter the value of the sum Z but will not affect its limit.

If an arc of a curve can be divided into a finite number of parts satisfying the conditions stated above regarding the arc AB, then its hyperbolic length is the sum of the hyperbolic lengths of its parts. For instance, we divide the arc AD shown in Fig. 12 into the parts AB, BC and CD, but we may mark the points of division on the arc CD, moving from D to C

Let the points of the half-plane τ be shifted in such a way that the hyperbolic length of any arc located on it is equal to the hyperbolic length of the same arc in its new position. We shall refer to this displacement of points as **hyperbolic**

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[1] The ratio of a chord to the arc it subtends is known to tend to unity if the length of the arc tends to zero (having in mind only the arcs of smooth curves)

motion, a concept similar to that of the motion of a Euclidean plane, e.g. to rotation of a Euclidean plane through a certain angle round one of its points.

If hyperbolic motion transforms a figure F into F_1, we call the figures F and F_1 hyperbolically equal.

Let us now consider the simplest hyperbolic motions.

(1) If each point on the half-plane τ is translated the same distance in one and the same direction parallel to a line u, then each figure is transformed into a figure hyperbolically equal to the initial one, since neither its Euclidean dimensions nor the distances of its points from u are altered.

Hence we conclude that the *Euclidean shift of the half-plane* τ *along a straight line is hyperbolic motion.*

(2) Let a similarity transformation with the similarity centre at an arbitrary point O on the line u and with a positive coefficient of similarity transform the segment MN into a segment M_1N_1 (Fig. 13). We denote the distances of the points N and N_1 from u by y and y_1. Because of the similarity of the triangles OMN and OM_1N_1 we will have $\frac{MN}{y} = \frac{M_1N_1}{y_1}$.

From this expression and Eq. (5) it follows that the hyperbolic length of any specific arc of any curve is not altered by this transformation.

Therefore, *similarity transformation with the similarity centre on a straight line u and with a positive coefficient of similarity is hyperbolic motion.*

The similarity coefficient is taken as positive so that the segment M_1N_1 will be in the half-plane τ and not in the half-plane τ'.

(3) Let us consider the inversion relative to a circle k of an arbitrary radius R with its centre O on a straight line u

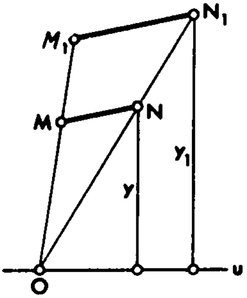

Fig 13

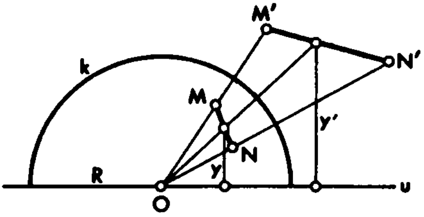

Fig. 14

(Fig. 14). Let M and N be two points sufficiently close to each other, and M' and N' two points symmetrical to them with respect to k. We denote by y and y' the distances from u of the points of intersection of the bisector of angle MON and the segments MN and $M'N'$. Since the triangles OMN and $OM'N'$ are similar,

$$\frac{MN}{y}=\frac{M'N'}{y'}$$

From this expression and Eq. (5) we conclude that the hyperbolic length of any specific arc of any curve is not altered by this transformation.

Consequently, *inversion with respect to a circle of any radius with its centre on a straight line is hyperbolic motion.*

(4) Finally, it is not hard to demonstrate that *a symmetry transformation with respect to an axis perpendicular to a straight line u is also hyperbolic motion.*

Each of the hyperbolic motions discussed, let us note, is a conformal transformation. This is obvious as regards the translation of the half-plane τ along the line u, and the similarity and symmetry transformations; as for the inversion, its conformality was demonstrated in Sec. 3.

Since hyperbolic motion has the property of transforming any figure into one hyperbolically equal to the first, any transformation consisting of a sequence of several hyperbolic motions has the same property, so that it too is hyperbolic motion.

We shall mention, without proof, that any hyperbolic motion can be presented as a sequence of a finite number of the simplest hyperbolic motions discussed above.

Let us now show that the laws of Lobachevskian geometry are realized in the half-plane τ if the rules for measuring lengths are as stated above.

We shall have to consider in the half-plane τ certain figures, characterized by the same properties as the corresponding figures of Euclidean geometry, but possibly differing from them in form; and we shall retain the terms of Euclidean geometry for them with the prefix "hyperbolic". For instance, we shall call the line, along which the shortest hyperbolic distance between any of its points is measured, a hyperbolic straight line; and we shall call the locus of points located at a specified hyperbolic distance from a given point, a hyperbolic circle.

Let us determine what curves in the half-plane τ are hyperbolic straight lines.

Hyperbolic straight lines will primarily be Euclidean rays perpendicular to the line u, as follows from the following argument.

Let points A and B lie on a perpendicular to a straight line u (Fig. 15) and be connected by the segment of the straight line AmB and by a curve or a broken line AnB. Let two straight lines a and b, parallel to u and sufficiently close to each other, cut AmB at points C and D, and the line AnB at points E and F. Since the Euclidean length of the segment CD, generally speaking, is shorter than that of the arc EF, and since their hyperbolic lengths can be considered equal to $\frac{CD}{y}$ and $\frac{EF}{y}$, where y is the distance from D (or F) to the line u, then the hyperbolic length of CD is, in general, smaller than that of the arc EF (these hyperbolic lengths can only be equal if EF is a segment of a Euclidean line perpendicular to u; it is clear that this condition is not always fulfilled since, in the opposite case, the arc AnB would coincide with

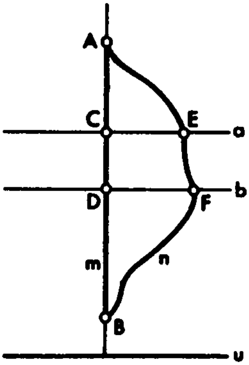

Fig. 15

the segment AmB). Hence, the hyperbolic length of the segment AmB is shorter than the hyperbolic length of the arc AnB. Q.E.D.

Let us show now that the semicircle of a Euclidean circle k with its centre on a straight line u is also a hyperbolic straight line.

Let k intersect u at points A and B (Fig. 16). Draw a circle q with its centre at A and assume it to be the circle of inversion. Let k and q intersect at points M and N. The circle k, passing through the pole of inversion, is transferred by inversion with respect to q into a straight line MN (see Sec. 3). Since inversion is a hyperbolic motion, and MN is perpendicular to u, it follows that the semicircle k is transformed by hyperbolic motion into a hyperbolic straight line. Therefore, this semicircle is a hyperbolic straight line as well.

Thus, hyperbolic straight lines of the half-plane τ are represented by Euclidean rays normal to the straight line u, and by Euclidean semicircles with centres on the line u. Later, in considering Axiom 1, we shall see that these are the only hyperbolic straight lines that exist.

Let us erect in the half-plane τ a perpendicular to the line u at an arbitrary point M in this line (Fig. 17), take a point A on it and construct points A_1, A_2, A_3, ... in such a way that the following equalities are satisfied:

$$AA_1 = A_1M, \quad A_1A_2 = A_2M, \quad A_2A_3 = A_3M, \quad \ldots$$

In other words, A_1 is the midpoint of the segment AM, A_2 is the midpoint of the segment A_1M, A_3 of the segment A_2M, and so on.

Let us consider the similarity transformation with the centre of similarity at M and a coefficient of similarity of 1/2. This

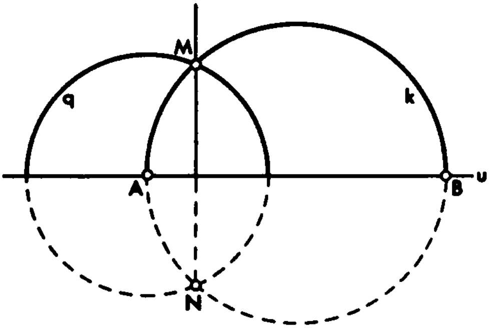

Fig 16

transformation is a hyperbolic motion transforming the points A, A_1, A_2, ... into the points A_1, A_2, A_3, ... respectively. Hence it follows that the hyperbolic lengths of the segments AA_1, A_1A_2, A_2A_3, ... are all equal to one another. Thus, the construction performed above comes down to laying off hyperbolically equal segments AA_1, A_1A_2, A_2A_3, ... from the point A on the hyperbolic straight line AM; and as will be seen from the construction, we never reach the point M however many segments we construct. Consequently, M is an infinitely removed point on the hyperbolic straight line AM. Since M is an arbitrary point on the line u, we conclude that each point on u is an infinitely removed point in the half-plane τ .

The process of laying off the hyperbolically equal segments AB_1, B_1B_2, B_2B_3, ... on the hyperbolic straight line AM (Fig. 17) can also be performed in the opposite direction to that used above, the process also being infinite. Hence it follows that a point on AM, infinitely removed in the sense of the Euclidean geometry, will at the same time be an infinitely removed point on the hyperbolic straight line AM.

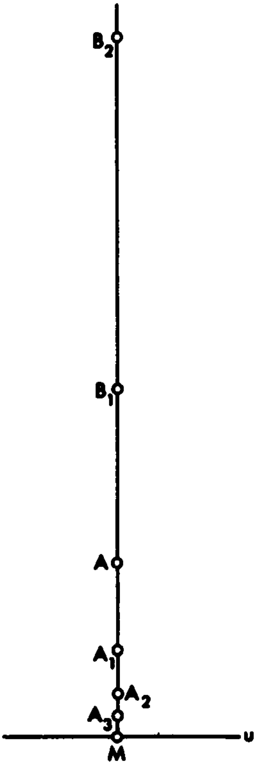

Fig. 17

Any point on the hyperbolic straight line AM, except the two points indicated above, will be at a finite hyperbolic distance from A, since at a sufficiently large finite value of the positive integer n it will be located either on the segment AA_n or on the segment AB_n.

Thus, the hyperbolic straight line AM, which means any hyperbolic straight line, has two, and only two, infinitely removed points.

If a hyperbolic straight line is represented by a Euclidean semicircle with its centre on the line u, then its points of intersection with u will be its infinitely removed points.

A Euclidean straight line, let us note, has only one infinitely removed point, which is the point in common of the given line and of all lines parallel to it.

Now it will readily be seen that all the plane axioms of Lobachevskian geometry hold in the half-plane τ. We shall limit ourselves to analysis of two.

Axiom 1. *One, and only one, hyperbolic straight line can be drawn through any two distinct points.*

If two given points A and B lie on a Euclidean perpendicular to a line u, this perpendicular will be the desired hyperbolic straight line. If they do not, we find on the line u a point N equidistant from A and B, and using it as the centre trace a Euclidean semicircle with radius NA (Fig. 18); this will be the desired hyperbolic straight line.

To demonstrate that two distinct hyperbolic straight lines l and l' cannot pass through two distinct points A and B, it is sufficient to assume that A and B lie on a Euclidean perpendicular l to a line u (Fig. 19), since any other case can be reduced to this by suitable hyperbolic motion. For such a configuration of points A and B the shortest hyperbolic distance between them, as shown above, is *only* that along the Euclidean straight line l and therefore l and l' coincide on the segment AB. Now let us assume that a point C, lying on l', is not on l, and that B lies on l' between A and C. Then the arc AC of a Euclidean semicircle k with centre on u is part of a hyperbolic straight line that does not coincide with l' on the segment AC; but that, as we have already seen, is impossible. Therefore, l and l' coincide everywhere.

Hence it follows that there are no other hyperbolic straight lines than Euclidean rays perpendicular to u, and Euclidean semicircles with their centres on u: one sole hyperbolic straight line passes through any two given points, and is one or other of these two types.

Axiom 2. *Two hyperbolic straight lines can be drawn parallel to a hyperbolic straight line p through a point P not on the line p.*

Two hyperbolic straight lines are called **parallel** if they have a common infinitely removed point. In particular, the hyperbolic straight lines represented by Euclidean perpendiculars to u, are parallel: their common point at infinity in the half-plane τ is identical to the one they share in the Euclidean plane ω.

Let us denote the infinitely removed points of a hyperbolic straight line p by A and B (Fig. 20) and draw through P and

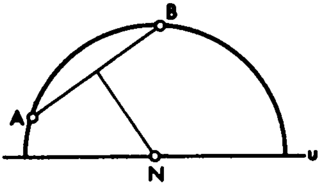

Fig. 18

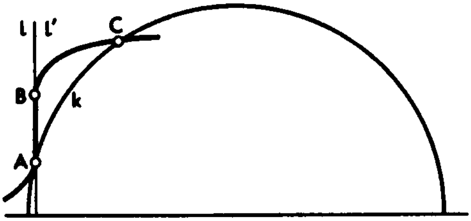

Fig. 19

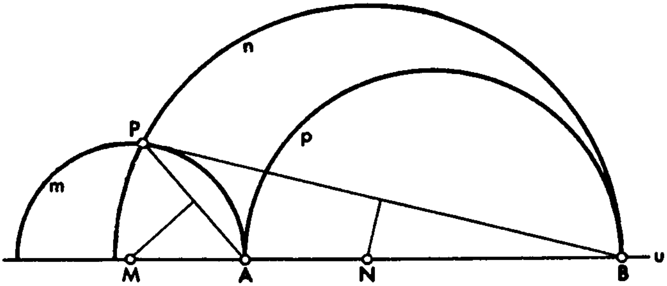

Fig 20

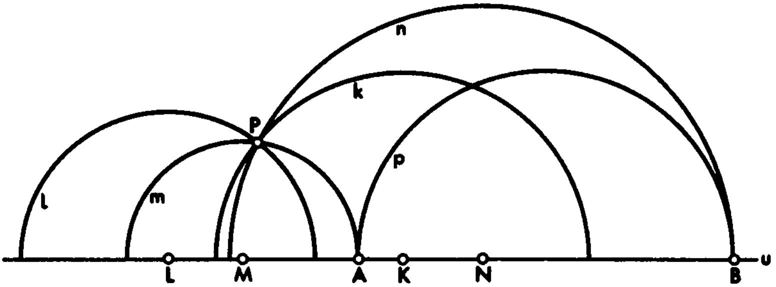

Fig. 21

A a Euclidean semicircle *m* with centre *M* on the line *u*, and through *P* and *B* a Euclidean semicircle *n* with centre *N* on the line *u*. These Euclidean semicircles will be the desired hyperbolic straight lines, being parallel to the hyperbolic straight line *p* in its different directions, viz. *m* in the direction from *B* to *A*, and *n* from *A* to *B*.

Three types of hyperbolic straight line pass through the point *P*: (1) intersecting the straight line *p*, (2) parallel to *p*, (3) not intersecting *p* and not parallel to it.

There are an infinite number of hyperbolic straight lines of the first type, and an infinite number of the third type, but only two of the second type.

To construct a hyperbolic straight line of the first type we take an arbitrary point *K* of a segment *MN* and trace a semicircle *k* with radius *KP* taking *K* as its centre (Fig. 21). If the same is done for an arbitrary point *L* on a line *u* outside the segment *MN* we obtain the third type of hyperbolic straight line *l* (see Fig. 21).

It will now be clear that Axiom 2 is equivalent to Lobachevsky's parallel-postulate formulated in Sec. 2.

If two hyperbolic straight lines neither intersect nor are parallel they are called **divergent**. The straight lines *p* and *l* (in Fig. 21), for example, diverge.

Thus, in the half-plane τ the axioms, and consequently the theorems, of Lobachevskian geometry are satisfied. Therefore the half-plane τ with the rules established above for measuring lengths is a Lobachevskian plane or, rather, a map of a Lobachevskian plane in a Euclidean plane.

It is instructive to compare this map with one of the Earth's surface in Mercator's projection; in the latter case meridians are projected as parallel lines perpendicular to the straight lines projecting geographic parallels (see Fig. 2 on p. 16). Great circles, meridians among them, must be considered "straight lines" on a sphere. Parallels, with the exception of the equator, are not "straight", though the map shows them as Euclidean straight lines. Similarly, among the Euclidean straight lines on the half-plane τ, perpendicular to the line *u* and parallel to it, the perpendiculars are hyperbolic straight lines and the parallels are not (this will be discussed in more detail in Sec. 7).

Furthermore, the length of a degree of a parallel is the shorter the higher the latitude, but in Mercator's projection the segment equalling 1° of a parallel has a constant length regardless of its latitude. The pattern is similar in the half-plane τ (see Principle 1°).

It is important to note that the τ-map is conformal, i. e. the Euclidean measure of an angle on this map is equal to its actual value in the Lobachevskian plane.

Let us demonstrate this first for a right angle. Inscribe a semicircle *k* with centre *M* on a line *u* and at *M* erect a ray *p* perpendicular to *u* (Fig. 22). Now consider the angles *1*, *2*, *3*, and *4* formed by the hyperbolic straight lines *k* and *p*. There is hyperbolic motion transforming the angles *1* into *2*, and *3* into *4* (symmetry with respect to *p*) and hyperbolic motion transforming *1* into *3* and *2* into *4* (inversion with respect to *k*). Hence, in the Lobachevskian plane (like in the τ-map) $\angle 1=\angle 2=\angle 3=\angle 4$; consequently, each of these angles is a right angle.

Employing the configuration used in Fig. 22, we denote the point of intersection of lines *k* and *p* by *A*, and one of the points of intersection of lines *k* and *u* by *N* (Fig. 23). A Euclidean semicircle *n* with centre *N* and radius *NA* will then divide $\angle 1$ in Fig. 22 into two angles 5 and 6, the Euclidean values of which, it will easily be seen, are equal. Inversion with respect to *n* transforms *k* into *p* and *p* into *k*, hence the angles *5* and *6* change places. Hence we conclude that not only the Euclidean values of $\angle 5$ and $\angle 6$ are equal but also their actual

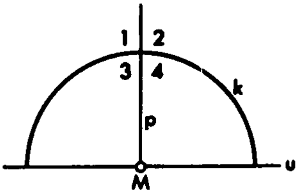

Fig 22

(hyperbolic) values, i. e. in the Lobachevskian plane (and on the τ-map) each is equal to half a right angle.

Let us denote by L the point of intersection of lines u and n, lying on the same side of the point M as the point N, and around L draw a circle l with radius LA (Fig. 23). The circle l will divide the angle 6 into the angles 7 and 8. It will readily be seen that

$$\angle 8 = \angle NAL = \frac{\pi}{8}$$

and since $\angle 6 = \frac{\pi}{4}$, then $\angle 7 = \frac{\pi}{8}$; hence the Euclidean values of the angles 7 and 8 are equal. At the same time their hyperbolic values are equal, since these angles are commuted by inversion with respect to l.

In the same way we prove that angles on the τ-map with Euclidean values $\frac{\pi}{16}, \frac{\pi}{32}$, ... have the same values in the Lobachevskian plane.

Since every angle can be represented as a sum of a finite number, or as the limit of a sum of an infinitely increasing number of terms of the form

$$\frac{\pi}{2},\quad \frac{\pi}{4},\quad \frac{\pi}{8},\quad \frac{\pi}{16},\quad \frac{\pi}{32}, \dots$$

the conformality of the τ-map is proved.

Section 5. A Circle in a Lobachevskian Plane

Let us determine how a circle in the Lobachevskian plane is represented on the τ-map.

Through a point M on the straight line u draw a Euclidean straight line p perpendicular to u, and take two arbitrary points B and C on it in the half-plane τ (Fig. 24; $MB > MC$). Now plot a point A on p such that the equality

$$\frac{CM}{AM} = \frac{AM}{BM} \tag{6}$$

is satisfied.

From this equality we deduce that the hyperbolic lengths of the segments CA and AB are identical. Indeed, the similarity transformation with similarity centre at M and a coefficient $\frac{CM}{AM}$ transforms the segment AB into CA[1].

---

[1] $BM \cdot \frac{CM}{AM} = BM \cdot \frac{AM}{BM} = AM$; consequently B is transformed into A; $AM \cdot \frac{CM}{AM} = CM$; consequently A is transformed into C.

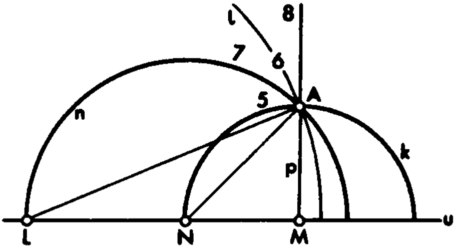

Fig. 23

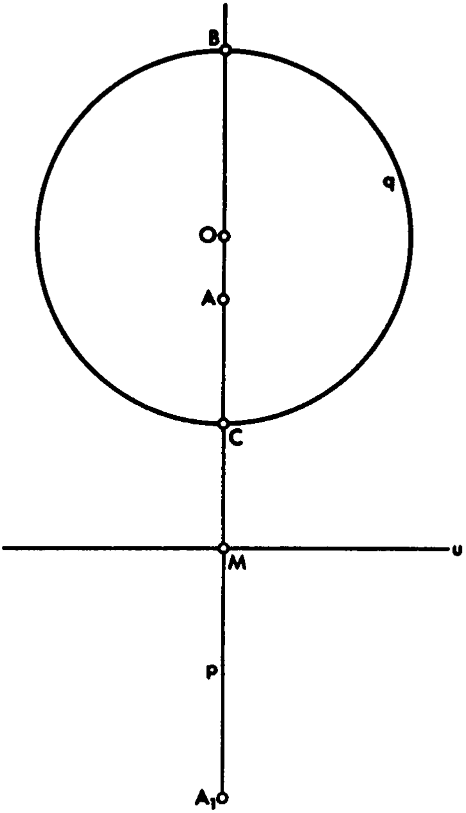

Fig. 24

Denoting the Euclidean midpoint of the segment BC by O, we trace a Euclidean circle q with centre O and radius OB, and plot a point A_1 symmetrical to A with respect to the line u.

Since

$$OA = OM - AM, \quad OA_1 = OM + MA_1 = OM + AM$$

then

$$OA \cdot OA_1 = OM^2 - AM^2 \tag{7}$$

Furthermore

$$OM = \frac{1}{2}(BM + CM)$$

and by virtue of equality (6)

$$AM^2 = BM \cdot CM$$

Therefore Eq. (7) can be presented in the form

$$OA \cdot OA_1 = \frac{1}{4}(BM + CM)^2 - BM \cdot CM =$$

$$= \frac{1}{4}(BM^2 + 2BM \cdot CM + CM^2 - 4BM \cdot CM)$$

or

$$OA \cdot OA_1 = \frac{1}{4}(BM - CM)^2 \tag{8}$$

Since

$$\frac{1}{2}(BM - CM) = OB$$

we obtain from Eq. (8)

$$OA \cdot OA_1 = OB^2$$

from which we deduce that the points A and A_1 are symmetrical with respect to the circle q.

Let us demonstrate that the hyperbolic distances from all points on the line q to the point A are equal.

Draw an arbitrary Euclidean circle n through A and A_1 (Fig. 25). Its centre N is on the line u, and, consequently, its portion lying in the half-plane τ is a hyperbolic straight line.

Let n and q intersect at points D and E, and n and u at points F and G. Draw a Euclidean circle f with centre F and radius FA. The circles q and f are mutually orthogonal since f passes through the points A and A_1 symmetrical with respect to q (see Sec. 3); therefore inversion with respect to f transforms the circle q into itself.

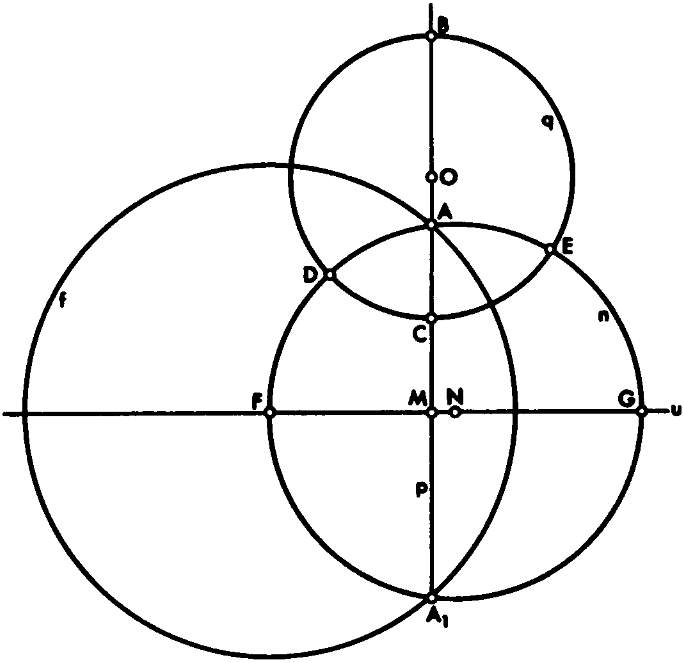

Fig. 25

Furthermore, the same inversion transforms the line *p*, which does not pass through the pole of inversion *F*, into a circle passing through *F*, as well as through the points *A* and A_1 (which remain fixed in this inversion), i. e. into the circle *n*. On the other hand, the circle *n*, which passes through the inversion pole, is transformed into a straight line, namely into *p*, since this line must pass through the points *A* and A_1.

Hence it follows that the arcs *AD* and *AE* of the circle *n* are transformed into the segments *AB* and *AC*, respectively, of the straight line *p*. Therefore, the hyperbolic lengths of the segments *AD* and *AE* of the hyperbolic straight line *n* are equal to the hyperbolic lengths of the segments *AB* and *AC* of the hyperbolic straight line *p*; in other words, the hyperbolic distances of the points *B*, *C*, *D*, and *E* from the point *A* are equal. This shows that a hyperbolic circle is projected onto the τ-map as a Euclidean circle with no points in common with the line *u*, but the projection of its centre (*A*) does not coincide with the centre (*O*) of the corresponding Euclidean circle.

In conclusion let us note that every hyperbolic straight line passing through A intersects the circle q at right angles, which is analogous to the known property of the diameters of a Euclidean circle.

Section 6. Equidistant Curve

Let p and q be a perpendicular and an oblique line meeting u at a point M, and P_1Q_1 and P_2Q_2 arcs of Euclidean circles with a common centre M, or, in other words, segments of two hyperbolic straight lines m_1 and m_2 (Fig. 26). Since m_1 and m_2 intersect p at right angles, the hyperbolic lengths of the arcs P_1Q_1 and P_2Q_2 give the hyperbolic distances of the points Q_1 and Q_2 from the hyperbolic straight line p, and these distances are equal since the arc P_1Q_1 can be transformed by similarity transformation with its centre at M into the arc P_2Q_2.

Hence we can conclude that the line q is the locus of points located at a specified hyperbolic distance from the hyperbolic straight line p. This line is called an **equidistant curve,** and the hyperbolic straight line p its **base**. An equidistant curve, as will be seen from Sec. 4, is not a hyperbolic straight line.

The assumption that the locus of points, spaced at one and the same distance from a given straight line and located on the same side of that line, is a straight line, contradicts this property of an equidistant curve and hence Lobachevsky's parallel-postulate, and is equivalent to the Euclidean parallel-postulate.

Note that hyperbolic straight lines perpendicular to the base of an equidistant curve cut it at right angles, as is evident from Fig. 26.

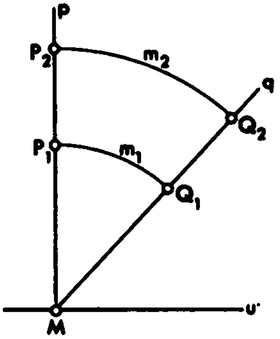

Fig. 26

Inversion with respect to a circle with its centre on a straight line *u* at a point distinct from *M*, transforms *q* into a Euclidean circle, which, like the hyperbolic straight line, intersects the line *u* but its centre is not on *u*.

Thus, an equidistant curve is represented on the τ-map either by a Euclidean ray cutting the line *u* at an acute or obtuse angle, or by an arc of a Euclidean circle cutting *u* but with its centre not on *u*. It will readily be seen that there is no other type of equidistant curve.

Section 7. Horocycle

Let us draw the diameter *p* of a circle *q*, perpendicular to a line *u*, denoting its point of intersection with *q*, nearest to *u*, by *C* (Fig. 27). If we retain the point *C* and increase the radius of the circle *q* without limit in such a way that its centre moves along the line *p* in the direction indicated by its arrow, then in the limiting configuration *q* will be transformed into a Euclidean straight line *h*, parallel to *u*.

The line *h* which is not a hyperbolic straight line, is called the **limit line,** or **horocycle.** Thus the limiting form of a circle, one of whose points and the tangent to that point are fixed, and whose radius increases without limit, is a straight line in Euclidean geometry, but a horocycle in Lobachevskian geometry.

Let us consider the hyperbolic motion represented by inversion with respect to a circle *n* with its centre *N* on a line *u* (Fig. 27). It transforms the line *h* into a Euclidean circle h_1

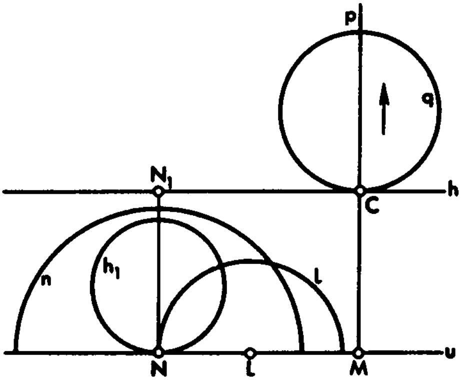

Fig. 27

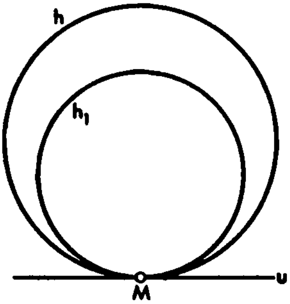

Fig. 28

passing through N and with its centre on the common perpendicular NN_1 of the Euclidean straight lines u and h; hence it follows that h_1 is a tangent to the line u.

Thus, an horocycle is represented on the τ-map either by a Euclidean straight line parallel to u, or by a Euclidean circle tangent to u.

Let us draw through N a Euclidean circle l with its centre L on the line u (Fig. 27). Since the radii of the Euclidean circles h_1 and l are mutually perpendicular, the hyperbolic straight line l intersects the horocycle h_1 at right angles. Hence we can conclude that all hyperbolic straight lines passing through an infinitely remote point on an horocycle (referred to as its **axes**) intersect the line at right angles.

Any horocycle h is hyperbolically equal to any other horocycle h_1, i. e. there is a hyperbolic motion that transforms h into h_1. These hyperbolic motions are: (*a*) similarity transformation with the similarity centre on the line u, when h and h_1 are Euclidean straight lines parallel to u, or Euclidean circles of different radii tangent to u (Figs. 28 and 29); (*b*) shift of the half-plane τ along the line u when h and h_1 are Euclidean circles of the same radius, tangent to u; (*c*) inversion with the pole on u when one of the curves h and h_1 is a Euclidean straight line parallel to u, and the other is a Euclidean circle tangent to u.

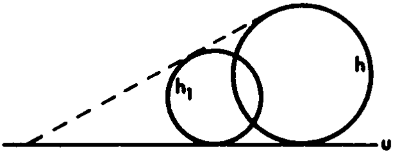

Fig 29

Section 8. Selected Theorems of Lobachevskian Geometry

Theorem 1. *The sum of the angles of any triangle is less than* π.

Let us first consider a right-angled triangle ABC (Fig. 30). Its sides a, b, c are represented respectively by a segment of a Euclidean perpendicular to the line u, an arc of a Euclidean circle with centre at M and an arc of a Euclidean circle with centre at N. The angle C is a right angle. The angle A is equal to the angle between the tangents to the circles b and c at the point A, or, which is the same thing, to the angle between the radii NA and MA of these circles. And, finally, $\angle B = \angle BNM$.

Let us draw a Euclidean circle q on the segment BN as a diameter; this circle will have only one point B in common with the circle c, since its diameter is the radius of the latter. Therefore the point A lies outside the circle bounded by q, whence

$$\angle A = \angle MAN < \angle MBN$$

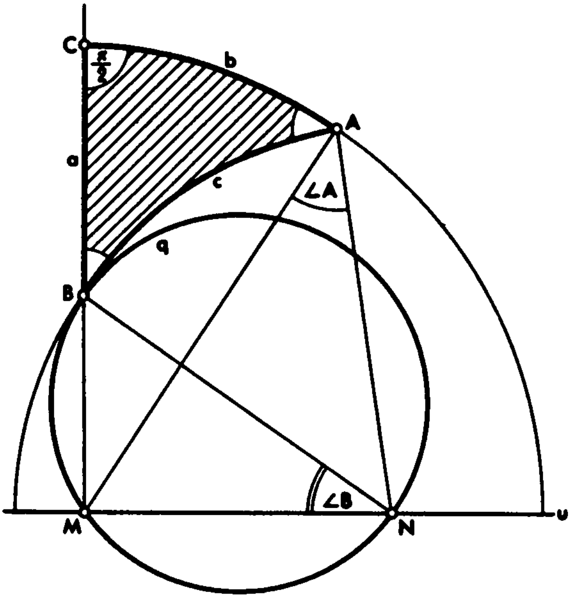

Fig. 30

Hence, owing to the relation $\angle MBN + \angle B = \pi/2$, we obtain

$$\angle A + \angle B < \pi/2 \tag{9}$$

and therefore $\angle A + \angle B + \angle C < \pi$. Q.E.D.

Any right-angled triangle, let us note, can be transferred by appropriate hyperbolic motion to a position in which one of its sides lies on a Euclidean perpendicular to the line u; consequently the method used above to derive inequality (9) is applicable to any right-angled triangle.

When a scalene triangle is given, we divide it by one of its altitudes into two right-angled triangles. The sum of the acute angles of these right-angled triangles is equal to the sum of angles of the given scalene triangle. From this, taking inequality (9) into account, we conclude that the theorem holds for any triangle.

Theorem 2. *The sum of the angles of a quadrangle is less than* 2π.

It is sufficient for proof to divide the quadrangle by a diagonal into two triangles.

Theorem 3. *Two divergent straight lines have one and only one perpendicular in common.*

Let one of the given divergent lines be represented on the τ-map by a Euclidean perpendicular p to the line u at the point M, and the other by a Euclidean semicircle q with its centre on u; and let p and q have no points in common (Fig. 31). Such a configuration of two divergent hyperbolic straight lines on the τ-map can always be achieved by means of the appropriate hyperbolic motion.

Let us draw from M a Euclidean tangent MN to q and construct a Euclidean semicircle m with centre M. It is clear that m is a hyperbolic straight line intersecting both p and q at right angles. Therefore, on the τ-map, m represents the desired common perpendicular to the given divergent straight lines.

Two divergent straight lines cannot have two common perpendi-

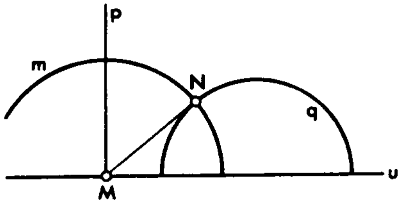

Fig 31

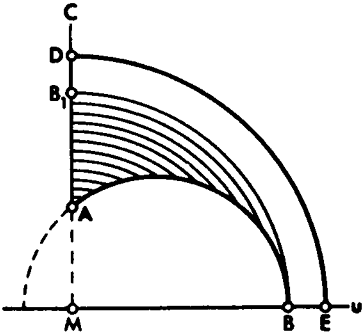

Fig. 32

culars since, in that case, a quadrangle could exist with four right angles, and that would contradict Theorem 2.

Theorem 4. *A rectangular projection of one side of an acute angle onto its other side is a segment* (and not a ray as in Euclidean geometry).

The validity of this theorem is evident from Fig. 32, in which the segment AB_1 is the rectangular projection of the side AB of the acute angle BAC on its side AC.

In the same figure the arc DE of the Euclidean circle with centre at M is a perpendicular to the hyperbolic straight line AC. This perpendicular does not meet the oblique line AB. Therefore the assumption that a perpendicular and an oblique line to the same straight line always intersect contradicts Lobachevsky's axiom of parallels, and is equivalent to Euclid's parallel-postulate.

Theorem 5. *If the three angles of a triangle ABC are equal, respectively, to the three angles of a triangle $A'B'C'$, then these triangles are equal.*

Assume the converse to be true and lay off the segments $AB_1 = A'B'$, $AC_1 = A'C'$ on the rays AB and AC respectively. It is obvious that triangles AB_1C_1 and $A'B'C'$ are equal, having two sides and the included angle equal. Since the point B_1 does not coincide with B, the point C_1 does not coincide with C, as otherwise the triangles would be equal, which contradicts the assumption.

Let us consider the following possibilities.

(a) The point B_1 lies between A and B, and the point C_1 between A and C (Fig. 33; both this figure and the next one arbitrarily present hyperbolic straight lines as Euclidean ones). It is easily seen that the sum of the angles of the quadrangle BCC_1B_1 is 2π, which is impossible according to Theorem 2.

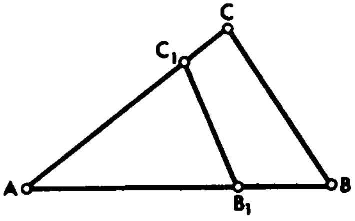

Fig 33

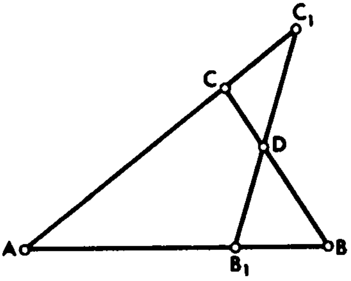

Fig. 34

(b) The point B_1 lies between A and B, and the point C between A and C_1 (Fig. 34). Let us denote the point of intersection of the segments BC and B_1C_1 by D. Since $\angle C = \angle C'$ and $\angle C' = \angle C_1$, then $\angle C = \angle C_1$, which is impossible since the angle C is exterior with respect to the triangle CC_1D.[1]

The other possible cases are interpreted in a similar way.

The theorem is proved since the assumption made above has led to a contradiction.

It follows from Theorem 5 that there is no triangle in Lobachevskian geometry that is similar to a given triangle but not equal to it.

Section 9. Supplementary Remarks

A number of important conclusions can be drawn from consideration of the τ-map.

First, each theorem of Lobachevskian geometry reduces on the τ-map to a certain theorem of Euclidean geometry. Therefore any contradiction in Lobachevskian geometry would necessarily lead to a contradiction in Euclidean geometry. Hence, Lobachevsky's geometry is consistent.

Second, acquaintance with Lobachevskian geometry enormously facilitates the detection of fallacies in attempts to prove Euclid's parallel-postulate, which boils down in most cases to assuming a proposition equivalent to this axiom. A proof that this assumption contradicts Lobachevsky's parallel-postulate is sufficient to invalidate it. That was the method we used in the three cases discussed above (viz. on the locus of points equidistant from a straight line; on the intersection of a perpendicular and an oblique line to a given line; on the existence of similar but not equal triangles).

---

[1] The proof of the theorem "an exterior angle of a triangle is greater than the interior angle not adjacent to it" does not depend on the parallel-postulate.

Let us consider another example. A mathematician of the last century, Farkas Bolyai (father of János Bolyai mentioned above), suggested a proof of Euclid's parallel-postulate based on the assumption that a circle can always be drawn through three points not on the same straight line. Bolyai considered this obvious, but in Lobachevskian geometry this is not the case since a circle, a horocycle, or an equidistant curve can pass through three points of the Lobachevskian plane not on the same straight line; consequently, a circle cannot always be drawn through three such points. Thus we see that Bolyai's assumption is equivalent to the Euclidean parallel-postulate which invalidates his proof.

Lobachevsky did not use the method of constructing a map of the hyperbolic plane in his own work; it was first suggested by the Italian geometer Eugenio Beltrami in a paper published in 1868, twelve years after the Russian geometer's death.

The map of the Lobachevskian plane discussed in this book differs considerably from that constructed by Beltrami, and was introduced by the French scientist Henri Poincaré (1854-1912).

Section 10. On Natural Logarithms and Hyperbolic Functions

The material presented below will be used in the sections that follow.

Let us first derive several important relations.

We introduce the notation:

$$a_n = \left(1+\frac{1}{n}\right)^n, \quad b_n = \left(1+\frac{1}{n}\right)^{n+1} \tag{10}$$

where n is a positive integer. Obviously

$$a_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}, \quad b_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+2} \tag{11}$$

From Eqs. (10) and (11) we obtain

$$b_n - a_n = \frac{1}{n}\left(1+\frac{1}{n}\right)^n = \frac{a_n}{n} \tag{12}$$

$$b_{n+1} - a_{n+1} = \frac{1}{n+1}\left(1+\frac{1}{n+1}\right)^{n+1} \tag{13}$$

and

$$b_n - a_{n+1} = \left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n+1}\right)^{n+1}$$

By factoring the right-hand side of the last equality we obtain

$$\frac{1}{n(n+1)}\left[\left(1+\frac{1}{n}\right)^{n} + \left(1+\frac{1}{n}\right)^{n-1}\left(1+\frac{1}{n+1}\right) + \dots \right.$$
$$\left. \dots + \left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)^{n-1} + \left(1+\frac{1}{n+1}\right)^{n}\right] \qquad (14)$$

Now by substituting $1+\frac{1}{n}$ for each cofactor $1+\frac{1}{n+1}$ in square brackets we increase expression (14), which, after simplification, yields

$$b_n - a_{n+1} < \frac{1}{n}\left(1+\frac{1}{n}\right)^{n}$$

From this, by means of Eq. (12), we obtain

$$b_n - a_{n+1} < b_n - a_n$$

or

$$a_{n+1} > a_n$$

Consequently a_n increases with the integer n.

Now let us substitute $1+\frac{1}{n+1}$ for each cofactor $1+\frac{1}{n}$ in square brackets of Eq. (14), which is thereby diminished; after simplification we obtain

$$b_n - a_{n+1} > \frac{1}{n}\left(1+\frac{1}{n+1}\right)^{n} \qquad (15)$$

It will easily be seen that

$$\frac{1}{n}\left(1+\frac{1}{n+1}\right)^{n} > \frac{1}{n+1}\left(1+\frac{1}{n+1}\right)^{n+1} \qquad (16)$$

indeed, after simplification we can reduce this expression either to

$$\frac{1}{n} > \frac{n+2}{(n+1)^2}$$

or to

$$(n+1)^2 > n(n+2)$$

This last inequality is obviously correct.

From (15), (16), and (13) we obtain

$$b_n - a_{n+1} > b_{n+1} - a_{n+1}$$

whence

$$b_n > b_{n+1}$$

Thus, b_n decreases as the integer n increases.

Since $a_1=2$ and $b_1=4$, we can conclude from this analysis that

$$2 \leqslant a_n < b_n \leqslant 4$$

From this inequality and from Eq. (12) it follows that

$$b_n - a_n < \frac{4}{n} \tag{17}$$

Since, as n increases, a_n increases, b_n diminishes and the difference $b_n - a_n$ tends to zero, as follows from inequality (17), the values a_n and b_n must tend to the same limit, which is usually denoted by the letter e, a_n always being less and b_n always greater than this limiting value. Thus

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} \tag{18}$$

and

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1} \tag{19}$$

In particular, with $n=1$ we have

$$2 < e < 4 \tag{20}$$

The number e is irrational; its approximate value is 2.71828.

From Eq. (19) follows an approximate equality

$$\left(1+\frac{1}{n}\right)^n \approx e \tag{21}$$

Its error is less than the difference $b_n - a_n$ and is consequently less than $\frac{4}{n}$

Let x be a positive rational proper fraction, and let us take such values of the positive integer n that $nx=k$ will be integer. Then by virtue of inequality (19) we obtain

$$\left(1+\frac{x}{k}\right)^k < e^x < \left(1+\frac{x}{k}\right)^{k+x}$$

Consequently, the following approximate equality will hold:

$$\left(1+\frac{x}{k}\right)^k \approx e^x \tag{22}$$

with an error less than

$$\left(1+\frac{x}{k}\right)^{k+x}-\left(1+\frac{x}{k}\right)^{k}=\left(1+\frac{x}{k}\right)^{k}\left[\left(1+\frac{x}{k}\right)^{x}-1\right]<\frac{xe^{x}}{k} \tag{23}$$

Furthermore, from the Newton binomial formula we have

$$\left(1+\frac{x}{k}\right)^{k}=1+x+\frac{k(k-1)}{2k^{2}}x^{2}+$$

$$+\frac{k(k-1)(k-2)}{6k^{3}}x^{3}+\ldots+\frac{1}{k^{k}}x^{k} \tag{24}$$

from which follows the approximate relation

$$\left(1+\frac{x}{k}\right)^{k}\approx 1+x \tag{25}$$

We denote its error by σ. It is obvious that

$$\sigma=\frac{x^{2}}{2}\left[\frac{k-1}{k}+\frac{(k-1)(k-2)}{3k^{2}}x+\ldots+\frac{2}{k^{k}}x^{k-2}\right]<$$

$$<\frac{x^{2}}{2}(1+x+x^{2}+\ldots)=\frac{x^{2}}{2(1-x)} \tag{26}$$

We conclude from (22), (25) and (26) that

$$e^{x}\approx 1+x \tag{27}$$

The error of this relation does not exceed $\frac{x^{2}}{2(1-x)}$, since the limit of the expression $\frac{xe^{x}}{k}$ (see Eq. (23)) is zero when k increases infinitely. This error can be made as small as desirable if x is given sufficiently small values.

Eq. (27) is also valid in the case when $x<1$ is a positive irrational number, as can be demonstrated if rational approximations of its value are considered.

It should be noted that Eq. (27) is also valid for negative values of x with absolute values less than unity; the error in this case does not exceed $\frac{x^{2}}{2(1+x)}$.

From Eqs. (22) and (24) we can obtain an additional approximate relation, more accurate than Eq. (27). Since $k\to\infty$, the limit of the third term on the right-hand side of Eq. (24) is $\frac{1}{2}x^{2}$. Consequently we can obtain

$$e^{x}\approx 1+x+\frac{1}{2}x^{2} \tag{28}$$

This formula is employed when x is so small that x^{3} can be neglected. We refrain from estimating the error of formula (28).

Let us now consider the system of logarithms with base e, or **natural logarithms**, which play so important a role in mathematics.

The natural logarithm of a number x is designated as $\ln x$. By virtue of the known properties of logarithms $\ln 1 = 0$, and $\ln e = 1$.

Taking the logarithm of both sides of Eq. (27) we obtain the following approximation:

$$\ln(1+x) \approx x \tag{29}$$

which can be employed when x is sufficiently small.

Hyperbolic functions, i. e. hyperbolic sines and cosines (designations sinh and cosh), are defined by means of the number e:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2} \tag{30}$$

Two other hyperbolic functions, the hyperbolic tangent and the hyperbolic cotangent (designations tanh and coth), can be defined as follows:

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x} \tag{31}$$

Hyperbolic functions possess several properties similar to those of the corresponding trigonometric functions.

The following approximate formulas are derived from (27), (30), and (31) for sufficiently small values of x:

$$\sinh x \approx x, \ \cosh x \approx 1, \ \tanh x \approx x \tag{32}$$

and from (28), (30), and (31)

$$\sinh x \approx x, \ \cosh x \approx 1 + \frac{1}{2}x^2, \ \tanh x \approx \frac{2x}{2+x^2} \tag{33}$$

Section 11. Measurement of Segments of Hyperbolic Straight Lines

In this section we shall show how the hyperbolic lengths of segments of hyperbolic straight lines are calculated.

Let us first consider a Euclidean ray in the half-plane τ. perpendicular to a line u at its point M (Fig. 35), and on it the points A, B, C, D spaced in such a way that

$$\frac{MB}{MA} = \frac{MD}{MC}$$

or, which is the same thing,

$$\frac{MB}{MD} = \frac{MA}{MC}$$

Denoting each of these two relations by μ we note that the similarity transformation with its centre at M and a coefficient μ transforms the segment CD into the segment AB; therefore, the hyperbolic lengths of these segments are equal.

From that it follows that the hyperbolic length of the segment AB (which we shall denote by AB_h) is characterized by the ratio $\frac{MB}{MA}$, or, in other words, is a function of this ratio. Let us show that the logarithm can be chosen for this function, i. e. that we can assume

$$AB_h = \log \frac{MB}{MA} \tag{34}$$

Let F be a point on the segment AB. Then

$$\frac{MB}{MA} = \frac{MF}{MA} \cdot \frac{MB}{MF}$$

Taking the logarithm of this relation we obtain, from Eq. (34),

$$AB_h = AF_h + FB_h$$

which corresponds to the rule of addition of segments.

Generally speaking, we can take the logarithm in Eq. (34) with any—but the same for all segments—positive base (other than unity); however, to match the rule in question with the contents of Sec. 4 we must restrict our choice to the natural logarithm and consequently write Eq. (34) in the form

$$AB_h = \ln \frac{MB}{MA} \tag{35}$$

Indeed, when the segment AB is sufficiently small compared with the segment MA, then, from the relations

$$\ln \frac{MB}{MA} = \ln \frac{MA + AB}{MA} = \ln\left(1 + \frac{AB}{MA}\right)$$

we obtain, by virtue of Eqs. (29) and (35),

$$AB_h \approx \frac{AB}{MA}$$

which corresponds to the principle assumed in Sec. 4.

It should be noted that the hyperbolic lengths of the segments AB and BA, calculated by means of Eq. (35), are equal in absolute value but have opposite signs. This demonstrates that reversing the direction of a segment alters the sign of its hyperbolic length. If the direction of the segment is of no interest to us, then the right-hand side of Eq. (35) should contain the absolute value of the logarithm.

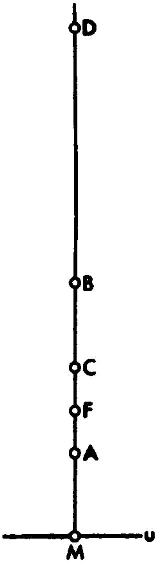

Fig. 35

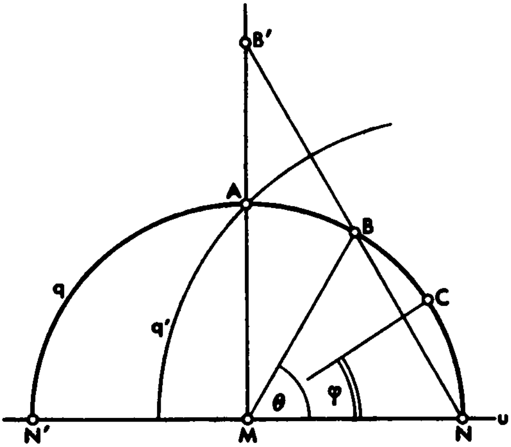

Fig. 36

Now we consider a Euclidean semicircle *q* with centre *M* on a line *u* and cutting *u* at points *N'* and *N*, and a Euclidean perpendicular to *u* at *M*, cutting *q* at the point *A* (Fig. 36).

Let *B* be a point on the arc *AN*. We draw a Euclidean straight line *NB* and denote its intersection with *MA* by *B'*. It is not hard to see that the segments *AB* and *AB'* of the hyperbolic straight lines *q* and *MA* are equal. Indeed, inversion with respect to the circle *q'* (with radius *NA* and centre at *N*) transforms *q* into a Euclidean straight line *MA*; the point *A* is thereby transformed into itself and the point *B* into *B'*, since both *B* and *B'* lie on a Euclidean straight line passing through the pole of inversion *N*. Therefore

$$AB_h = AB'_h = \ln \frac{MB'}{MA}$$

We denote the angle *NMB* by θ; then $\angle MNB = 90° - \frac{\theta}{2}$ and

$$\frac{MB'}{MA} = \frac{MB'}{MN} = \tan\left(90° - \frac{\theta}{2}\right) = \cot\frac{\theta}{2}$$

Hence

$$AB_h = \ln \cot\frac{\theta}{2} \tag{36}$$

If C is a point on the arc BN (Fig. 36) and $\angle NMC = \varphi$ then, as follows from Eq. (36),

$$AC_h = \ln \cot \frac{\varphi}{2}, \quad BC_h = AC_h - AB_h = \ln \cot \frac{\varphi}{2} - \ln \cot \frac{\theta}{2}$$

Hence

$$BC_h = \ln\left(\cot \frac{\varphi}{2} \cdot \tan \frac{\theta}{2}\right) \tag{37}$$

We have thus obtained the formulas both for the case when a hyperbolic straight line containing a given segment is represented by a Euclidean ray, and for the case when it is represented by a Euclidean semicircle.

Section 12. Basic Formulas of Hyperbolic Trigonometry

Let us consider a right-angled triangle ABC in the half-plane τ (Fig. 37). Its side BC is a segment of the Euclidean straight line OB ($OB \perp u$), the side CA is an arc of a Euclidean circle with radius 1 and centre O, the side AB is an arc of a Euclidean circle with radius l and centre M; $\angle C$ is a right angle, $\angle A = \alpha$, and $\angle B = \beta$.

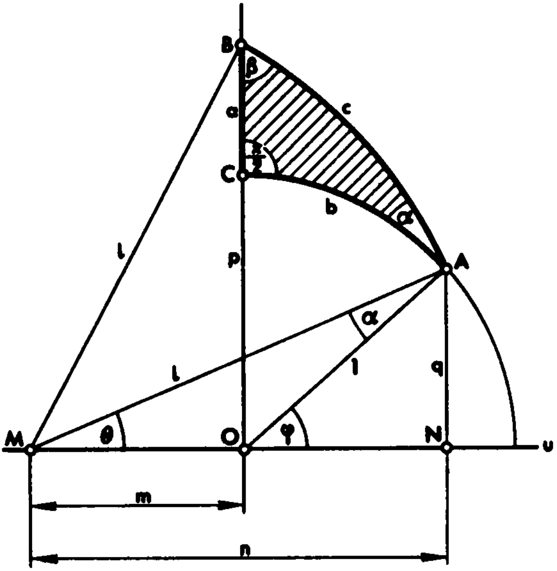

Fig. 37

We drop a perpendicular AN from the point A onto the line u and introduce the notation

$$OB = p, \quad NA = q, \quad MO = m, \quad MN = n, \quad \angle NMA = \theta, \quad \angle NOA = \varphi$$

We denote the *hyperbolic* lengths of the sides BC, CA and AB of the given triangle by a, b and c respectively. (Conversely, l, m, n, p, q are the *Euclidean* lengths.)

Note that

$$\angle OAM = \alpha, \quad \angle OMB = \beta$$

since the tangents to the sides of the angle A at A are perpendicular to the sides of the angle OAM, and the tangents at B to the sides of the angle B are perpendicular to the sides of the angle OMB.

Let us now establish a number of relations between the quantities in question

From the triangles OBM and OAM we obtain

$$p^2 = l^2 - m^2$$

$$1 = l^2 + m^2 - 2mn (= OA^2)$$

Hence

$$p^2 - 1 = 2m(n - m), \quad p^2 + 1 = 2(l^2 - mn) \tag{38}$$

Furthermore, by virtue of Eq. (35),

$$a = \ln \frac{p}{1} = \ln p$$

Consequently

$$e^a = p, \quad e^{-a} = \frac{1}{p}$$

$$\sinh a = \frac{1}{2}(e^a - e^{-a}) = \frac{1}{2}\left(p - \frac{1}{p}\right) = \frac{p^2 - 1}{2p}$$

$$\cosh a = \frac{1}{2}(e^a + e^{-a}) = \frac{1}{2}\left(p + \frac{1}{p}\right) = \frac{p^2 + 1}{2p}$$

Hence, employing Eqs. (38), we obtain

$$\sinh a = \frac{m(n - m)}{p}, \quad \cosh a = \frac{l^2 - mn}{p} \tag{39}$$

From the triangle OAN we have

$$\sin \varphi = q, \quad \cos \varphi = n - m \tag{40}$$

Consequently

$$\cot\frac{\varphi}{2}=\frac{1+\cos\varphi}{\sin\varphi}=\frac{1+n-m}{q}$$

$$\tan\frac{\varphi}{2}=\frac{1-\cos\varphi}{\sin\varphi}=\frac{1-n+m}{q}$$

Since, from Eq. (36),

$$b=\ln\cot\frac{\varphi}{2}$$

then

$$e^{b}=\cot\frac{\varphi}{2}=\frac{1+n-m}{q},\qquad e^{-b}=\tan\frac{\varphi}{2}=\frac{1-n+m}{q}$$

Hence

$$\sinh b=\frac{n-m}{q},\qquad \cosh b=\frac{1}{q} \tag{41}$$

Furthermore, from the triangles *OBM* and *OAN* we obtain

$$\sin\theta=\frac{q}{l},\qquad \cos\theta=\frac{n}{l} \tag{42}$$

$$\sin\beta=\frac{p}{l},\qquad \cos\beta=\frac{m}{l} \tag{43}$$

Hence

$$\cot\frac{\theta}{2}=\frac{1+\cos\theta}{\sin\theta}=\frac{l+n}{q},\qquad \tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}=\frac{l-n}{q}$$

$$\tan\frac{\beta}{2}=\frac{1-\cos\beta}{\sin\beta}=\frac{l-m}{p},\qquad \cot\frac{\beta}{2}=\frac{1+\cos\beta}{\sin\beta}=\frac{l+m}{p}$$

Since, from Eq. (37),

$$c=\left(\ln\cot\frac{\theta}{2}\tan\frac{\beta}{2}\right)$$

then

$$e^{c}=\cot\frac{\theta}{2}\tan\frac{\beta}{2}=\frac{(l+n)(l-m)}{pq}=\frac{l^2+ln-lm-mn}{pq}$$

$$e^{-c}=\tan\frac{\theta}{2}\cot\frac{\beta}{2}=\frac{(l-n)(l+m)}{pq}=\frac{l^2-ln+lm-mn}{pq}$$

Therefore

$$\sinh c = \frac{l(n-m)}{pq}, \quad \cosh c = \frac{l^2 - mn}{pq} \tag{44}$$

Finally, from the triangle *OAM* we obtain

$$\alpha = \varphi - \theta$$

Taking Eqs. (40) and (42) into account, we derive:

$$\sin\alpha = \sin\varphi\cos\theta - \cos\varphi\sin\theta = \frac{qn - q(n-m)}{l}$$

$$\cos\alpha = \cos\varphi\cos\theta + \sin\varphi\sin\theta = \frac{n(n-m) + q^2}{l} = \frac{n(n-m) + l^2 - n^2}{l}$$

since $q^2 = l^2 - n^2$. Thus,

$$\sin\alpha = \frac{qm}{l}, \quad \cos\alpha = \frac{l^2 - mn}{l} \tag{45}$$

From Eqs. (39), (41), (43), (44) and (45) we obtain

$$\tanh a = \frac{m(n-m)}{l^2 - mn}, \quad \tanh b = n - m, \quad \tanh c = \frac{l(n-m)}{l^2 - mn} \tag{46}$$

$$\tan\alpha = \frac{qm}{l^2 - mn}, \quad \cot\alpha = \frac{l^2 - mn}{qm} \tag{47}$$

$$\tan\beta = \frac{p}{m}, \quad \cot\beta = \frac{m}{p} \tag{48}$$

It is not difficult, by means of Eqs. (39), (41), and (43)-(48), to verify the validity of the following formulas which constitute the basic formulas of hyperbolic trigonometry:

$$\cosh c = \cosh a \cdot \cosh b \tag{49}$$
$$\sinh a = \sinh c \cdot \sin\alpha \tag{50}$$
$$\sinh b = \sinh c \cdot \sin\beta \tag{51}$$
$$\tanh a = \sinh b \cdot \tan\alpha \tag{52}$$
$$\tanh b = \sinh a \cdot \tan\beta \tag{53}$$
$$\tanh a = \tanh c \cdot \cos\beta \tag{54}$$
$$\tanh b = \tanh c \cdot \cos\alpha \tag{55}$$
$$\cos\alpha = \cosh a \cdot \sin\beta \tag{56}$$
$$\cos\beta = \cosh b \cdot \sin\alpha \tag{57}$$
$$\cosh c = \cot\alpha \cdot \cot\beta \tag{58}$$

Eqs. (49)-(58) can be rephrased into more general form if the quantities a, b, c in them are replaced by $\frac{a}{r}$, $\frac{b}{r}$, and $\frac{c}{r}$ respectively, which is equivalent to altering the scale of the hyperbolic lengths. Here r is a constant common to all segments.

It is typical that approximate relations, similar to the formulas of Euclidean trigonometry, can be derived from the foregoing relations between the elements of a right-angled triangle, given sufficiently small values of a, b, and c. For instance, by employing Eqs. (32) and (33), we obtain the following from Eqs. (50), (52) and (54):

$$a \approx c \sin \alpha$$
$$a \approx b \tan \alpha$$
$$a \approx c \cos \beta$$

transform Eq. (49) to

$$1+\frac{1}{2} c^2 \approx \left(1+\frac{1}{2} a^2\right)\left(1+\frac{1}{2} b^2\right)$$

whence we obtain

$$\frac{1}{2} c^2 \approx \frac{1}{2} a^2+\frac{1}{2} b^2+\frac{1}{4} a^2 b^2$$

After simplification, and neglecting the last term on the right-hand side because of its smallness, we arrive at

$$c^2 \approx a^2+b^2$$

Thus formula (49) corresponds to Pythagoras' theorem in Euclidean geometry.

Section 13. The Lengths of Certain Plane Curves in Lobachevskian Geometry

Length of an arc of a horocycle. In Fig. 38 the arc ADB of the Euclidean circle with centre O on a line u represents a segment of a hyperbolic straight line, and the Euclidean segment AB, parallel to u, represents an arc of the horocycle

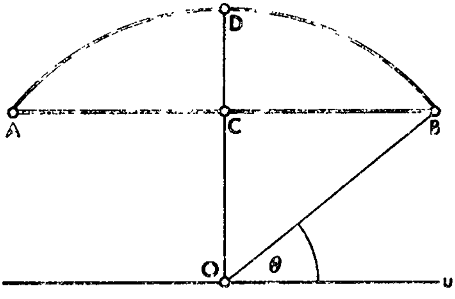

Fig 38

the hyperbolic lengths of which we denote by $2a$ and $2s$ respectively

Using Eq. (36) we obtain $a = \ln \cot \frac{\theta}{2}$, whence follows $\cot \frac{\theta}{2} = e^a$. Further, application of Principle 1° (Sec. 4) leads to·

$$s = \frac{AC}{OC} = \cot \theta = \frac{1}{2}\left(\cot \frac{\theta}{2} - \tan \frac{\theta}{2}\right) = \frac{1}{2}(e^a - e^{-a})$$

From the definition of the hyperbolic sine we obtain

$$s = \sinh a \qquad (59)$$

consequently, $2s = 2 \sinh a$. Thus, the length of an arc of the horocycle is double the hyperbolic sine of half of the chord subtending this arc.

Since $a < s$, we find from Eq. (59) that

$$a < \sinh a \quad (\text{if } a > 0) \qquad (60)$$

Length of a circle. As a preliminary step, we prove two auxiliary propositions

(a) If a is sufficiently small, then $\tanh a < a$.[1] Indeed, from Eq. (33) we have

$$\tanh a \approx \frac{2a}{2+a^2} < a \quad (\text{if } a > 0)$$

(b) Taking into account that the perimeters of regular n-sided polygons, both inscribed in a Euclidean circle with radius 1 and circumscribed about it, tend, with infinitely increasing integer n, to one and the same limit equal to the length of this circle, we obtain

$$\lim_{n \to \infty} 2n \sin \frac{\pi}{n} = \lim_{n \to \infty} 2n \tan \frac{\pi}{n} = 2\pi \qquad (61)$$

Let us now find the length s of a hyperbolic circle of radius R. (All symbols here and below refer to hyperbolic lengths.) Let AB and CD be the sides of regular n-sided polygons inscribed in this circle[2] and circumscribed about it. We

---

[1] We mention, without proof, that this inequality is valid for any positive value of a.

[2] Let A be a point on a hyperbolic circle q with centre O Construct an angle $AOM = \frac{\pi}{m}$, in which m is a given positive integer, and draw a tangent to the circle q at the point A This tangent and the ray OM will either meet at a certain point B, or will not intersect. In the first case the segment AB will be half the side of a regular m-sided

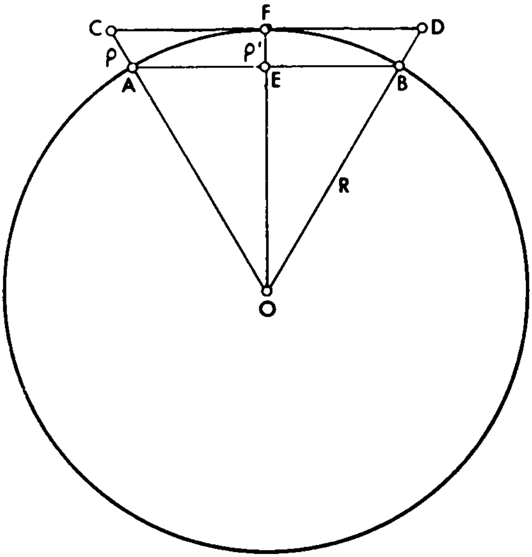

Fig 39

denote their circumferences by p and P, and the lengths of the segments AC and EF by ρ and ρ′ (see Fig. 39; hyperbolic figures are represented in it arbitrarily as Euclidean ones).

From Eqs. (52) and (50) we obtain for the right-angled triangles OAE and OCF, in which O is the centre of the given circle,

$$\tanh AE = \sinh OE \cdot \tan \frac{\pi}{n}$$

$$\sinh CF = \sinh OC \cdot \sin \frac{\pi}{n}$$

or

$$\tanh \frac{p}{2n} = \sinh (R - \rho') \cdot \tan \frac{\pi}{n} \tag{62}$$

$$\sinh \frac{P}{2n} = \sinh (R + \rho) \cdot \sin \frac{\pi}{n} \tag{63}$$

Let n be so large that $\tanh \frac{p}{2n} < \frac{p}{2n}$; since, according to

---

polygon circumscribed about the circle q. In the second case no regular m-gon can be circumscribed about q, though a regular n-gon can, provided the integer n, greater than m, is sufficiently large.

inequality (60), $\frac{P}{2n} < \sinh\frac{P}{2n}$, by multiplying Eqs. (62) and (63) term by term by $2n$, we obtain

$$\sinh(R-\rho')\cdot 2n\tan\frac{\pi}{n} < p < s < P < \sinh(R+\rho)\cdot 2n\sin\frac{\pi}{n} \qquad (64)$$

Taking equalities (61) into account and noting that ρ and ρ' tend to zero when n increases without limit, we come to the conclusion that the first and last terms in the series of inequalities (64) tend to one and the same limit $2\pi\sinh R$, coinciding with the quantity s:

$$s = 2\pi\sinh R$$

Thus, the length of a circle in Lobachevskian geometry is equal to the hyperbolic sine of its radius multiplied by 2π.

The length of an arc of an equidistant curve. Let P_1, P_2, ..., P_{n-1} be points located at Euclidean distances y_1, y_2, ..., y_{n-1} from a line u, dividing the segment AB into n equal parts (in the Euclidean sense), and let the Euclidean lengths of segments OB and AB be equal to y_n and ζ (Fig. 40; $OB \perp u$). Consider the arcs AA', P_1P_1', ..., BB' of Euclidean circles with

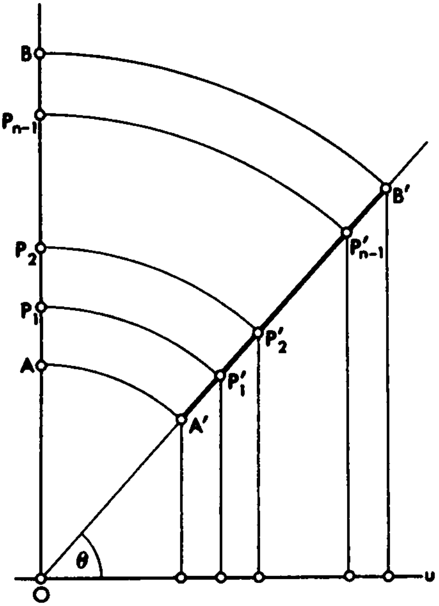

Fig 40

a common centre O, representing perpendiculars dropped from points of the equidistant OB' onto its base OB. The hyperbolic length h of each of these perpendiculars is determined, according to Eq. (36), by the relation $h = \ln \cot \frac{\theta}{2}$.

Let us denote the hyperbolic lengths of the arc $A'B'$ of a given equidistant curve and of the segment AB of its base by s and a. Since the Euclidean distances of the points P'_1, P'_2, .., B' from the line u are equal, respectively, to $y_1 \sin\theta$, $y_2 \sin\theta$, ..., $y_n \sin\theta$, and the Euclidean length of each part into which the segments AB and $A'B'$ are divided, is equal to $\frac{\zeta}{n}$, then, by virtue of the conclusions of Sec. 4, we shall obtain $a = \lim\limits_{n\to\infty} Z$, $s = \lim\limits_{n\to\infty} Z'$, where

$$Z = \frac{\zeta}{n}\left(\frac{1}{y_1} + \frac{1}{y_2} + \dots + \frac{1}{y_n}\right)$$

$$Z' = \frac{\zeta}{n}\left(\frac{1}{y_1 \sin\theta} + \frac{1}{y_2 \sin\theta} + \dots + \frac{1}{y_n \sin\theta}\right)$$

Hence

$$\frac{Z'}{Z} = \frac{1}{\sin\theta}$$

Since the ratio of the quantities Z' and Z remains constant, the ratio of their limits will have the same value:

$$\frac{s}{a} = \frac{1}{\sin\theta} = \frac{1}{2}\left(\cot\frac{\theta}{2} + \tan\frac{\theta}{2}\right) = \frac{1}{2}(e^h + e^{-h}) = \cosh h$$

Therefore

$$s = a \cosh h$$

Thus, the length of an arc of an equidistant curve is equal to the rectangular projection of the arc on the base of the equidistant multiplied by the hyperbolic cosine of the distance between its points and the base.

Conclusion

In concluding our booklet we would like to acquaint readers (without going into the proofs) with certain propositions of Lobachevskian geometry that bring out its special character.

First we shall describe a surface of Euclidean space that was mentioned only in passing in Sec. 2.

In Fig. 41 we have a Euclidean plane in which are a straight line *a* and a related curve *t* (**tractrix**) with the property that at each point on it the segment of the tangent to *t*, contained between the tangency point of the tangent and its point of intersection with the line *a*, is constant in length irrespective of the choice of the tangency point.

If the tractrix *t* is rotated around the straight line *a*, it forms a surface known as a **pseudosphere** (Fig. 42).

The pseudosphere is the surface investigated by Beltrami, who proved that it has properties inherent in a segment of a Lobachevskian plane (if the shortest paths on this surface are considered "straight").

Similarly, there is a surface in Lobachevskian space in which (given the same interpretation of "straight line") the propositions of Euclidean geometry for a plane are satisfied, a surface known as the **horosphere**; it is generated by a horocycle rotating around one of its axes.

Let us now formulate several of the simplest propositions characteristic of Lobachevskian geometry.

1. Two parallel straight lines approach each other asymptotically in the direction of their parallelism (i. e. the distance

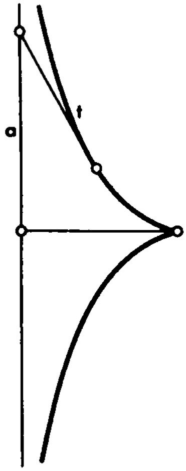

Fig. 41

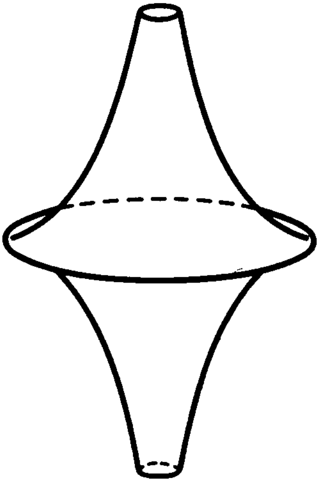

Fig. 42

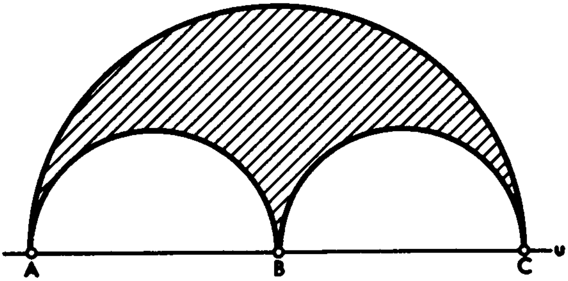

Fig. 43

between a point on one of them and on the other can be made as small as specified) and diverge without limit in the opposite direction.

2. Let a straight line *c* intersect two divergent straight lines *a* and *b* at points *A* and *B*. The length of the segment *AB* will be minimal when *c* coincides with the common perpendicular of the given divergent straight lines (*a* and *b* diverge without limit on both sides of their common perpendicular).

3. The area of a triangle *ABC* is $r^2 (\pi - \angle A - \angle B - \angle C)$ where the angles are measured in radians and *r* is the constant mentioned in Sec. 12 common for all triangles. The triangle with all three angles equal to zero will have the maximal area πr^2 (such a triangle is shown hatched in Fig. 43).

4. An angle inscribed in a circle is not always measured by half the arc on which it is based. In particular, the angle based on the diameter is always acute (and not a right angle as in Euclidean geometry).

5. Given an arbitrary integer *n*, $n > 6$, we can construct a circle such that the side of a regular *n*-sided polygon inscribed in it is equal to its radius. The side of a regular hexagon inscribed in a circle is always greater than its radius.

6. In certain cases Lobachevskian geometry makes it possible to square a circle, i. e. to construct, using only a ruler and compasses, an equivalent circle and "square" (more precisely, an equiangular rhomb, since no quadrangle with four right angles can exist in a hyperbolic plane). In Euclidean geometry squaring a circle, of course, is impossible.

The examples given above indicate how wide the discrepancies between the conclusions of the two geometries can sometimes be.

* * *

We have traced only a few of the milestones on the road leading into the depths of hyperbolic geometry. We shall be happy if the reader introduced to the principles of this remarkable science by our exposition grows interested in it and wants to study it further in the special treatises devoted to it, including the work of its founder N. I. Lobachevsky.

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